



Winter Braids Lecture Notes

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Heegaard Floer Homologies and Rational Cuspidal Curves. Lecture notes.

ADAM BARANOWSKI, MACIEJ BORODZIK AND JUAN SERRANO DE RODRIGO

Abstract

This is an expanded version of the lecture course the second author gave at Winterbraids VI in Lille in February 2016.

1. Introduction

Heegaard Floer homologies were defined around 2000 by Ozsváth and Szabó. Since then a lot of research has been done in the subject and the number of papers that have appeared in the last 15 years is immense. It appears now that the whole knot theory and topology of three-manifolds has been affected at least in some way by this new theory.

Even though it is generally believed and almost completely proved (see [43]) that Heegaard Floer theory contains the same amount of information as the Seiberg–Witten theory, the Heegaard Floer theory has an advantage over the latter, namely often problems in Heegaard Floer theory can be reduced to combinatorics of Heegaard diagrams, which makes Heegaard Floer theory more accessible to an inexperienced reader. Moreover, this combinatorial flavor of Heegaard Floer theory sometimes makes it possible to effectively calculate Heegaard Floer homology groups, for example from a surgery formula [91, 92, 58].

As for the knot Floer theory: given any knot, there is not only an algorithm calculating knot homology groups [78], but also one often understands general properties of Floer chain complexes for knots, like torus knots and alternating knots, including two-bridge knots.

The immense speed of the development of Heegaard Floer theory makes it quite difficult for a non-expert to get an overview of the field. In the ever-growing pile of articles on the subject it might be hard not to get lost and to find the most important articles. Luckily, a few excellent survey papers have appeared: those by Ozsváth and Szabó [89, 90], and more modern ones of Juhász [34] and Manolescu [55]. A recent book [78] covers the grid diagram approaches to Heegaard Floer theory.

The aim of these notes is to give another introduction into the subject but this time with a clear view towards algebraic geometry. We focus on parts of the theory which are relevant in the applications, like L-space knots and d -invariants. We omit parts which, at least at present, have little application in algebraic geometry.

1.1. What is not in the notes?

Actually only a small part of the theory is covered in the notes. We do not mention any analytic difficulties with defining the Heegaard Floer theory rigorously, like compactness and smoothness of the moduli space of holomorphic disks used in [82]. We focus mostly on

rational homology three–spheres, not mentioning technical issues with defining Heegaard Floer homologies on manifolds with $b_1(Y) > 0$. In particular, we do not discuss the action of $\Lambda^* H_1(Y; \mathbb{Z})$ on the Heegaard Floer chain complex. Refer to [45] for more details.

Knot Floer homology is defined via Heegaard diagrams and only for knots. In the notes we do not give any construction via grid diagrams, even though it is purely combinatorial and has much less prerequisite knowledge; nonetheless it seems somehow that the original approach of Rasmussen and Ozsváth–Szabó reveals better why knot Floer homology is such a powerful tool. For a detailed account on grid Floer homology we refer to an excellent book of Ozsváth, Stipsicz and Szabó [78] mentioned above. For other ways to define the knot Floer homology we refer to the survey of Manolescu [55] and references therein.

We do not discuss the construction and properties of Heegaard Floer theory for links. The definition might seem very similar for links as it is for knots, yet the applications are much harder. In particular, the surgery formula for links is very hard, see [58] for details and [48] for exemplary applications.

We do not introduce the τ -invariant, which is a smooth concordance invariant that detects the four-genus of many knots, including torus knots, see [81]: for algebraic links it is equal to the three-genus anyway, so it does not bring any new piece of information about algebraic knots. Likewise, we do not discuss the Υ function of Ozsváth, Stipsicz and Szabó [77], which is a significant refinement of the τ invariant. For algebraic knots the Υ function is related to the V_m invariants; see [5].

Concordance invariants are only mentioned in the paper, we refer to a recent survey of Hom [30] for more details. The whole research concerning alternating links and Heegaard Floer–thin links is not mentioned in the article; see [80, 57]. We do not discuss double branched covers of links and their d -invariants, like in [56]. We do not provide any relations of Heegaard Floer theory with Khovanov homology; like in [80].

Sutured Heegaard Floer theory [33] as well as its younger cousin, the bordered Floer theory, see [51, 52, 53], is not covered in these notes. Bordered Heegaard Floer theory is a generalization of the Heegaard Floer theory for three-manifolds with boundary, with the aim to calculate Heegaard Floer homology groups by a cut-and-paste method. The algebraic setup for the bordered Floer theory is rather complicated, but the theory itself contains a lot of information, for example the S -equivalence class of a Seifert matrix of a knot can be read off from the bordered Floer homology of the knot complement, see [31]. It is known that knot Floer homology does not determine the Seifert matrix, see the discussion in [31, Section 1].

On the singularity theory side, we do not give full details on the classification of algebraic knots (and links). A concise but self-contained description is given in the book of Eisenbud–Neumann [14], which is also very well suited for topologists. We discuss only quickly and superficially the theory of rational cuspidal curves, referring to the thesis of Moe [62] or to a book of Namba [67] for a more classical version. The techniques such as spectrum semicontinuity or applications of the Bogomolov–Miyaoka–Yau inequality are not given. A reader wishing to learn methods of spectrum semicontinuity is referred to [16], a nice application of the Bogomolov–Miyaoka–Yau inequality in the theory of rational cuspidal curves is given also in [76].

1.2. What is in the notes?

Compared to what is not in the notes, the content of the paper is very scarce. With a view towards applications in algebraic geometry we try to give just about enough details for the reader to understand the two results about semigroup distribution property of rational cuspidal curves: Theorem 7.13 and Theorem 7.14, as well as their proofs. Consequently, we introduce Heegaard Floer homology in Section 2, where we also give a very brief description of Spin^c structures on three- and four-manifolds. In Section 3 we state two main results on

Heegaard Floer theory: the adjunction inequality and the surgery exact sequence. These results are used in proofs of most of the main theorems on Heegaard Floer theory. Section 4 deals with cobordisms in Heegaard Floer theory, in particular, we define d -invariants, show their behavior on the cobordism and define the absolute grading in the Heegaard Floer homology. At present, Theorem 4.6 is the most important result of Ozsváth–Szabó from the point of view of applications in algebraic geometry.

Next we discuss knot Floer homology in Section 5. Our emphasis is on the V_m -invariants for knots introduced in Section 5.4 and then on L-space knots, which we discuss in detail in Section 5.6.

Section 6 contains a (short and by no means complete) account on cuspidal singularities. We give basic definitions and pass quickly to the construction and basic properties of semigroups of singular points. We provide relations between semigroups and Alexander polynomials. We finish by linking the semigroups of singular points with the V_m -invariants of the links of singularities.

In Section 7 we first go quickly through recent results on rational cuspidal curves and give Theorem 7.13 and 7.14, which are central results of these notes. We then discuss a relation of these results with the FLMN conjecture (Conjecture 7.17), whose motivation we also recall. Finally, we show highlights and weak points of Theorem 7.14, as well as a counterexample to the original Conjecture 7.17 found by Bodnár and Némethi.

We have decided to give the reader a lot of problems to solve. Most of these are quick observations, some of them might require extra work. There is one problem, namely Problem 73, which is a research problem.

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2. Heegaard Floer homology

2.1. Preliminaries. Spin^c structures on three- and four-manifolds.

This section gathers some facts about Spin^c structures, which will be used in later sections. We will consider only Spin^c structures on the tangent bundle of a manifold. We refer to [19, Chapter 2] for a more detailed discussion. Other, concise references are [21, Section 1.4] or [75, Section 1.3]. A reader might want to skip this section at first reading.

Recall that for $n \geq 3$ the fundamental group of the special orthogonal group $\text{SO}(n) := \text{SO}(n, \mathbb{R})$ is $\pi_1(\text{SO}(n)) = \mathbb{Z}_2$. We define the *spin group* $\text{Spin}(n)$ to be the non-trivial double cover of $\text{SO}(n)$, thus in the case $n \geq 3$, it is the universal cover of $\text{SO}(n)$. By the construction there is a canonical inclusion $\mathbb{Z}_2 \hookrightarrow \text{Spin}(n)$. The group $\text{Spin}^c(n)$ is defined to be

$$(2.1) \quad \text{Spin}^c(n) := (\text{Spin}(n) \times \text{U}(1))/\mathbb{Z}_2.$$

It fits into the following short exact sequence

$$1 \rightarrow \text{U}(1) \xrightarrow{i} \text{Spin}^c(n) \xrightarrow{p} \text{SO}(n) \rightarrow 1,$$

where i sends z to $[1, z]$ and p is the projection of $\text{Spin}^c(n)$ onto $\text{SO}(n)$ via $\text{Spin}(n)$.

Problem 1. Verify that the projection p is well defined and gives rise to the short exact sequence above.

Consider now an oriented, n -dimensional Riemannian manifold M . We can regard the tangent bundle TM as associated to the $\text{SO}(n)$ -principal bundle $P_{\text{SO}(n)}$ of oriented orthonormal frames.

Definition 2.1 (*Spin^c structure*). A *Spin^c structure* on M is a pair (P, Λ) consisting of a $\text{Spin}^c(n)$ -principal bundle P over M and a map $\Lambda : P \rightarrow P_{SO(n)}$ such that the diagram

$$\begin{array}{ccc}
 \text{Spin}^c(n) \times P & \longrightarrow & P \\
 \downarrow p \times \Lambda & & \downarrow \Lambda \\
 SO(n) \times P_{SO(n)} & \longrightarrow & P_{SO(n)}
 \end{array}
 \begin{array}{c}
 \nearrow \\
 \searrow \\
 M
 \end{array}$$

with horizontal maps being the group actions on principal bundles, commutes. We denote the set of all Spin^c structures on M as $\text{Spin}^c(M)$.

There is a group homomorphism $\pi : \text{Spin}^c(n) \rightarrow S^1$, a projection on the second factor in (2.1) given by $[g, z] \mapsto z^2$. The composition $\pi \circ i$ is then a double cover of S^1 . Thus, given a Spin^c structure (P, Λ) on M , the map π can be used to construct an S^1 -principal bundle $P_1 = P/\text{Spin}(n)$ over M . From this we can define the so-called *determinant line bundle* $L \rightarrow M$, which is given by $L = P_1 \times_{S^1} \mathbb{C}$. One can in fact think of a Spin^c structure on M as of a choice of a complex line bundle L and a Spin structure on $TM \otimes L^{-1}$. We refer to [75, Section 1.3] and [19, Section 2.4] for more details.

Definition 2.2. The *first Chern class* of a Spin^c structure \mathfrak{s} on a manifold M is $c_1(\mathfrak{s}) = c_1(L)$.

As $TM \otimes L^{-1}$ is a Spin bundle, its second Stiefel–Whitney class vanishes. A quick calculus on characteristic classes yields the following fact, see [75, Section 1.3.3].

Proposition 2.3. We have that $c_1(\mathfrak{s}) \bmod 2 \equiv w_2(M)$, where $w_2(M)$ is the second Stiefel–Whitney class of M .

Remark 2.4. The meaning of ‘mod 2’ can be made precise by considering the short exact sequence $0 \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \rightarrow \mathbb{Z}_2 \rightarrow 0$. Associated with it is a long cohomology exact sequence (this is best seen, when using Čech homology, see [75]), in particular there is a well-defined map $H^j(X; \mathbb{Z}) \rightarrow H^j(X; \mathbb{Z}_2)$ for any compact topological space X and any $j \geq 0$. This map is often denoted $x \mapsto x \bmod 2$.

Proposition 2.5 (see [75, Proposition 1.3.14, Exercise 1.3.12]). *Let M be a closed oriented manifold. Let $\mathcal{L}_M \subset H^2(M; \mathbb{Z})$ be the set of integral lifts of the second Stiefel–Whitney class $w_2(M)$. The map $c_1 : \text{Spin}^c(M) \rightarrow \mathcal{L}_M$ is surjective. Moreover, if $H^2(M; \mathbb{Z})$ has no 2-torsion, then this map is also injective.*

Problem 2. Show that if M is simply connected, then $H^2(M; \mathbb{Z})$ has no 2-torsion.

Definition 2.6. An element $x \in \mathcal{L}_M$ is called a *characteristic element*.

In other words for manifolds such that $H^2(M; \mathbb{Z})$ has no 2-torsion, Spin^c structures correspond precisely to characteristic elements.

Another way of understanding Spin^c structures on a manifold is to see that two different Spin^c structures on M differ by a complex line bundle, hence the class of isomorphisms of complex line bundles (which in the smooth category is the same as $H^2(M; \mathbb{Z})$) acts on the set of all Spin^c structures on M . This action can be shown to be transitive and free, see again [75, Section 1.3], however there is (usually) no canonical identification of $\text{Spin}^c(M)$ with $H^2(M; \mathbb{Z})$. Anyway, if $H^2(M; \mathbb{Z})$ is finite, then the number of Spin^c structures on M is equal to the cardinality of $H^2(M; \mathbb{Z})$.

We also recall another equivalent formulation of Spin^c structures on three-manifolds due to Turaev [109]. Let M be a closed, connected, oriented three-manifold. An *Euler structure* is an equivalence class of non-vanishing vector field on M , where two vector fields v and w are said to be equivalent if there exists a closed ball $B \subset M$ such that v is homotopic to w through non-vanishing vector fields on $M \setminus \text{Int } B$.

Proposition 2.7 (see [109]). *The set of Euler structures on a three-manifold is in a one-to-one correspondence with the set of Spin^c structures.*

Problem 3. Construct geometrically a transitive and free action of $H_1(M; \mathbb{Z})$ on the set of all Euler structures on a closed three-manifold.

We pass to a description of Spin^c structures on four-manifolds. We begin with the following fact.

Lemma 2.8 (see [21, Proposition 1.4.18]). *Let M be a four-manifold with the intersection form $Q: H_2(M; \mathbb{Z}) \times H_2(M; \mathbb{Z}) \rightarrow \mathbb{Z}$. Then for any $x \in H_2(M; \mathbb{Z})$ we have $\langle w_2(M), x \rangle \equiv Q(x, x) \pmod{2}$.*

Corollary 2.9. *If M is a closed simply-connected four-manifold, then Spin^c structures on M are in a one-to-one correspondence with the elements $K \in H^2(M; \mathbb{Z})$ such that $Q(x, x) \equiv \langle K, x \rangle \pmod{2}$ for all $x \in H_2(M; \mathbb{Z})$.*

2.2. Heegaard diagrams

A genus g handlebody U is the boundary connected sum of g copies of a solid torus $D^2 \times S^1$. In other words, it is a three-manifold diffeomorphic to a regular neighborhood of a bouquet of g circles in \mathbb{R}^3 . The boundary of U is an oriented surface Σ of genus g .

Definition 2.10 (Heegaard decomposition). Let M be a closed, oriented, connected three-manifold. A *Heegaard decomposition* is a presentation of M as a union $U_0 \cup_{\Sigma} U_1$, where U_0 and U_1 are handlebodies and Σ is a closed, connected surface.

Problem 4. Show that the only manifold admitting a Heegaard decomposition of genus 0 is S^3 .

Example 2.11. If U_0 and U_1 are two solid tori glued along their boundary, then M is either S^3 , $S^2 \times S^1$ or a lens space.

To see this, denote by m_i and l_i the meridian and the longitude of the solid torus U_i , $i = 1, 2$. In order to glue the two tori we need to determine which curve on the torus ∂U_0 will be the meridian of ∂U_1 , that is, $m_1 = pm_0 + ql_0$ for some $p, q \in \mathbb{Z}$. Since m_1 is a closed curve, $\gcd(p, q) = 1$. We consider two cases: if $q = 0$, then $p = 1$, and we identify m_0 with m_1 and l_0 with l_1 . The resulting three-manifold is $S^2 \times S^1$. For the case $q \neq 0$ we will show that the construction above is equivalent to the usual construction of a lens space defined as the quotient of S^3 by an action of \mathbb{Z}_q . In order to do that, consider S^3 as a subset of \mathbb{C}^2 obtained by gluing two solid tori $U_0 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1, |z_1|^2 \geq \frac{1}{2} \geq |z_2|^2\}$, $U_1 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1, |z_2|^2 \geq \frac{1}{2} \geq |z_1|^2\}$ along the torus $\Sigma = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 = |z_2|^2 = \frac{1}{2}\}$. Observe that each of these sets is preserved by an action of \mathbb{Z}_q given by $[1] \cdot (z_1, z_2) = (e^{2\pi i/q} \cdot z_1, e^{2\pi i p/q} \cdot z_2)$, and the orbits $U_0/\mathbb{Z}_q, U_1/\mathbb{Z}_q$ are again solid tori. Finally, the quotient Σ/\mathbb{Z}_q is a torus. Upon closer examination of the way these two quotient tori are glued under this action, one may notice that the meridian m_1 of U_1/\mathbb{Z}_q is mapped exactly to the curve $pm_0 + ql_0$ on U_0/\mathbb{Z}_q ; see e.g. [101] for the details.

Theorem 2.12. *Each three-manifold M admits a Heegaard decomposition.*

Sketch of proof. Let $F: M \rightarrow [0, 3]$ be a self-indexing Morse function, that is, a Morse function such that the critical levels of index k are all at the level set $F^{-1}(k)$. (Such a function exists by [60].) Using an argument of [60], we might and actually will assume that F has only one minimum and only one maximum. Define $U_0 = F^{-1}[0, 3/2]$, $U_1 = F^{-1}[3/2, 3]$ and $\Sigma = F^{-1}(3/2)$. As F has only one minimum and one maximum, all of the three spaces U_0, U_1 and Σ are connected. In particular, Σ is a closed connected surface. The genus $g(\Sigma)$ is equal to the number of critical points F of index 1. By construction, U_0 and U_1 are genus g handlebodies. This shows the existence of a Heegaard decomposition. \square

A Heegaard decomposition is definitely not unique. One of the methods of obtaining a new Heegaard decomposition from another one is the following.

Given a Heegaard decomposition $M = U_0 \cup_{\Sigma} U_1$ of genus g , choose two points in Σ and connect them by an unknotted arc γ in U_1 . Let U'_0 be the union of U_0 and a small tubular neighborhood N of γ . Similarly, let $U'_1 = U_1 \setminus N$. The new decomposition $M = U'_0 \cup_{\Sigma'} U'_1$ is called the *stabilization* of $M = U_0 \cup_{\Sigma} U_1$. Clearly $g(\Sigma') = g(\Sigma) + 1$. Stabilizations and destabilizations will be discussed in a greater detail below (see Theorem 2.17).

In fact any two Heegaard decompositions are related by stabilizations and destabilizations (a precise statement is given in Theorem 2.17 below). This can be seen using Cerf theory [12]. Any two Morse functions F_0 and F_1 on M can be connected by a path F_t , $t \in [0, 1]$ in the space of all smooth functions from M to \mathbb{R} in such a way that for all but finitely many values $t \in [0, 1]$, F_t is a Morse function and there is a finite number of special values t_1, \dots, t_n at which a cancellation or a creation of a pair of critical points occurs. A more detailed analysis reveals that stabilizations and destabilizations of Heegaard diagrams correspond to creations, respectively, cancellations, of pairs of critical points of index 1 and 2. We omit the details, referring to [12]. An interested reader might find helpful a detailed exposition of the subject in [35].

Problem 5. Construct explicitly a Heegaard decomposition of S^3 of an arbitrary genus g .

Theorem 2.12 allows us to think of a three-manifold Y as a pair of two handlebodies U_0 and U_1 glued along their boundaries via a homeomorphism $\phi: \partial U_0 \rightarrow \partial U_1$. As isotopic homeomorphisms ϕ give rise to homeomorphic manifolds, in general, ϕ is an element of the mapping class group of ∂U_0 , and elements in mapping class groups are rather hard to deal with. Luckily, there is a more geometric point of view of a Heegaard decomposition. Suppose that F is a Morse function on Y such that $F^{-1}[0, 3/2] = U_0$ and $F^{-1}[3/2, 3] = U_1$. Let $\Sigma = F^{-1}(3/2) = \partial U_0 = \partial U_1$. Choose a Riemannian metric on M and consider the gradient ∇F . Critical points of F correspond to stationary points of the vector field ∇F and the Morse condition means that the stationary points are hyperbolic, hence the stable and unstable manifolds are well defined. (We refer to [23] for more details on stable and unstable manifolds.) Moreover, the Morse index of F gives precise information about the dimensions of the stable and unstable manifolds given by the stationary points of ∇F . Each index 1 critical point of F has a 2-dimensional unstable manifold of ∇F . Likewise, each index 2 critical point of F has a 2-dimensional stable manifold of ∇F . The unstable manifold of a critical point of index 1 intersects Σ along a simple closed curve and the stable manifold of a critical point of index 2 intersects Σ along a simple closed curve.

If the genus of the Heegaard decomposition is g , the above procedure yields precisely g simple closed curves on Σ obtained as intersections of unstable manifolds of critical points of index 1 with Σ , and g simple closed curves obtained as intersections of stable manifolds of critical points of index 2 with Σ . Call the first set of curves $\alpha_1, \dots, \alpha_g$ and the second set β_1, \dots, β_g . We will often call these curves α -curves and β -curves. By construction both the α -curves and the β -curves are pairwise disjoint. If ∇F satisfies the Morse–Smale condition, then the α -curves intersect the β -curves transversally.

Problem 6. Prove that each of the α curves constructed above is homologically trivial in U_0 and each of the β -curves is homologically trivial in U_1 .

Show even more, namely, that the curves $\alpha_1, \dots, \alpha_g$ span $\ker H_1(\Sigma; \mathbb{Z}) \rightarrow H_1(U_0; \mathbb{Z})$ and that a similar statement holds for the β -curves.

The last problem leads to the following definition:

Definition 2.13 (Heegaard diagram). Let $Y = U_0 \cup_{\Sigma} U_1$ be a Heegaard decomposition of a three-manifold Y , and let g be the genus of Σ . A *Heegaard diagram* is a triple (Σ, α, β) , where α and β are unordered collections of g simple closed curves $\alpha_1, \dots, \alpha_g$ and β_1, \dots, β_g , such that:

- $\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = \emptyset$ if $i \neq j$.
- The curves $\{\alpha_1, \dots, \alpha_g\}$ form a basis of $\ker(H_1(\Sigma; \mathbb{Z}) \rightarrow H_1(U_0; \mathbb{Z}))$ and $\{\beta_1, \dots, \beta_g\}$ form a basis of $\ker(H_1(\Sigma; \mathbb{Z}) \rightarrow H_1(U_1; \mathbb{Z}))$.

Problem 7. Consider $\Sigma \times [0, 1]$. Thicken all the α -curves on $\Sigma \times \{1\}$ to obtain pairwise disjoint annuli $A_1, \dots, A_g \subset \Sigma \times \{1\}$. Set $A = A_1 \cup \dots \cup A_g$. Define

$$H = \Sigma \times [0, 1] \cup_A \bigcup_{j=1}^g D_j,$$

where $D_j = D \times I$ is a disk D cross the interval I , glued to $\Sigma \times \{1\}$ along A_j . Prove that ∂H is a disjoint union of $\Sigma \times \{0\}$ and a two-sphere.

Use this problem to explicitly reconstruct a three-manifold from Σ and the collection of α - and β -curves.

The approach to three-manifolds via Heegaard diagrams allows us to obtain a combinatorial approach to studying three-manifolds. Heegaard Floer theory can be regarded as a way of extracting information about the three-manifold from the combinatorics of a Heegaard diagram.

Before we go further, we need to understand how a Heegaard diagram depends on the choice of the Morse function F .

Remark 2.14. If the Heegaard diagram is built from the vector field ∇F , that is, the α -curves and the β -curves are the intersections of the unstable and stable manifolds with Σ , then the Heegaard diagram depends also on the Riemannian metric used to define the vector field ∇F .

Definition 2.15. Two Heegaard diagrams (Σ, α, β) , $(\Sigma', \alpha', \beta')$ are *diffeomorphic* if there is an orientation-preserving diffeomorphism of Σ to Σ' that carries α to α' and β to β' .

Definition 2.16 (Handlesliding). Let U be a handlebody and denote by $\gamma = \{\gamma_1, \dots, \gamma_g\}$ a set of attaching circles for U . Let $\gamma_i, \gamma_j \in \gamma$ with $i \neq j$. We say that γ'_i is obtained from *handlesliding* γ_i over γ_j if γ'_i is any simple closed curve which is disjoint from the $\gamma_1, \dots, \gamma_g$, and the curves $\gamma'_i, \gamma_i, \gamma_j$ bound a pair of pants in Σ (see Figure 2.1). In that case, the set $\gamma' = \{\gamma_1, \dots, \gamma_{i-1}, \gamma'_i, \gamma_{i+1}, \dots, \gamma_g\}$ (with γ_i replaced by γ'_i) is also a set of attaching circles for U .

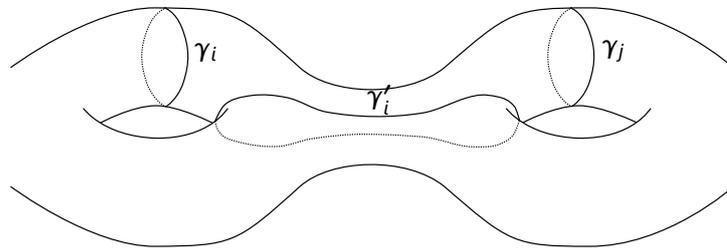


Figure 2.1: Handlesliding γ_i over γ_j .

The following result is classical, we refer to [82, Proposition 2.2] for a proof. One can find a detailed discussion in [35] as well.

Theorem 2.17. Two Heegaard diagrams (Σ, α, β) and $(\Sigma', \alpha', \beta')$ represent the same three-manifold if and only if they are diffeomorphic after a finite sequence of the following moves:

1. **Isotopy.** Two Heegaard diagrams (Σ, α, β) and $(\Sigma', \alpha', \beta')$ are isotopic if Σ and Σ' are of the same genus and there are two one-parameter families α_t and β_t of g -tuples of curves, moving by isotopies so that, for each t , both the α_t and the β_t are g -tuples of smoothly embedded, pairwise disjoint curves such that $(\alpha_0, \beta_0) = (\alpha, \beta)$, $(\alpha_1, \beta_1) = (\alpha', \beta')$.
2. **Stabilization.** We say that the diagram $(\Sigma', \alpha', \beta')$ is obtained from (Σ, α, β) by stabilization if $\Sigma' \cong \Sigma \# T^2$ (connected sum) and $\alpha' = \{\alpha_1, \dots, \alpha_g, \alpha_{g+1}\}$, $\beta' = \{\beta_1, \dots, \beta_g, \beta_{g+1}\}$, where $\alpha_{g+1}, \beta_{g+1}$ is a pair of curves in T^2 which meet transversally in a single point.
3. **Destabilization.** It is an inverse move to a stabilization.
4. **Handleslide.** We say that the diagram $(\Sigma', \alpha', \beta')$ is obtained from (Σ, α, β) by a handleslide if Σ and Σ' are of the same genus and either $\alpha = \alpha'$ and β' is obtained from β by a handleslide, or $\beta = \beta'$ and α' is obtained from α by a handleslide.

Idea of proof. One of the methods of proving, or at least understanding, the result, is to use Cerf theory again. Namely, choose a Riemannian metric on a three-manifold Y and suppose F_0 and F_1 are two different Morse functions on Y having a single minimum. We connect F_0 and F_1 by a generic path F_t in the space of smooth functions on Y as we did above. This time, however, we take into account not only situations, where F_t ceases to be a Morse function (which correspond to births/deaths of critical points), but also situations, where ∇F_t ceases to be a Morse–Smale flow. These situations correspond precisely to the handle slides. A very detailed discussion is included in [35]; the proof of Theorem 2.17 in [82] does not appeal to Cerf theory. \square

In Heegaard Floer theory, we will need to add an extra structure on Heegaard diagrams.

Definition 2.18 (Pointed Heegaard diagram). A *pointed Heegaard diagram* is a quadruple $(\Sigma, \alpha, \beta, z)$, where $z \in \Sigma \setminus (\alpha \cup \beta)$.

2.3. Symmetric products

Let $(\Sigma, \alpha, \beta, z)$ be a pointed Heegaard diagram. Let us consider the symmetric product

$$\text{Sym}^g(\Sigma) = \overbrace{\Sigma \times \dots \times \Sigma}^g / S_g,$$

where S_g is the symmetric group on g letters. In other words, $\text{Sym}^g(\Sigma)$ consists of unordered g -tuples of points in Σ where we also allow repeated points. Observe that $\text{Sym}^g(\Sigma)$ is a manifold.

Problem 8. Prove that $\pi_1(\text{Sym}^g(\Sigma))$ is abelian.

Problem 9. Let $i: H_1(\Sigma; \mathbb{Z}) \rightarrow H_1(\text{Sym}^g(\Sigma); \mathbb{Z})$ be a map induced by the inclusion $\Sigma \times \{*\} \times \dots \times \{*\}$ to $\text{Sym}^g(\Sigma)$. On the other hand, observe that a curve in $\text{Sym}^g(\Sigma)$ in a general position corresponds to a map from a g -fold cover of S^1 to Σ and in this way we might define a map $j: H_1(\text{Sym}^g(\Sigma); \mathbb{Z}) \rightarrow H_1(\Sigma; \mathbb{Z})$. Show that the two maps i and j are inverse to each other.

Proposition 2.19 (see [82, Proposition 2.7]). *Let $g > 2$, then $\pi_2(\text{Sym}^g(\Sigma)) = \mathbb{Z}$ and the action of $\pi_1(\text{Sym}^g(\Sigma))$ on $\pi_2(\text{Sym}^g(\Sigma))$ is trivial.*

Remark 2.20. For $g = 2$ we still have $\pi_2(\text{Sym}^g(\Sigma)) = \mathbb{Z}$, but the action of $\pi_1(\text{Sym}^g(\Sigma))$ is no longer trivial, this poses minor problems, one avoids them by requiring $g > 2$.

The manifold $\text{Sym}^g(\Sigma)$ inherits a complex structure and a symplectic structure from Σ . Let J denote this complex structure on $\text{Sym}^g(\Sigma)$. Consider the products

$$\mathbb{T}_\alpha = \alpha_1 \times \dots \times \alpha_g / S_g$$

and

$$\mathbb{T}_\beta = \beta_1 \times \dots \times \beta_g / S_g.$$

Problem 10. Show that \mathbb{T}_α and \mathbb{T}_β are totally real, that is, at each point $x \in \mathbb{T}_\alpha$ we have $T_x \mathbb{T}_\alpha \cap J T_x \mathbb{T}_\alpha = \{0\}$.

Remark 2.21. The fact that \mathbb{T}_α and \mathbb{T}_β are totally real might make one think that Heegaard Floer theory is a Lagrangian Floer theory of the intersections of \mathbb{T}_α and \mathbb{T}_β . While this was generally believed since the dawn of Heegaard Floer theory, the details were worked out only a few years later by Perutz [96].

Problem 11. Show that there is a 1–1 correspondence between points $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and g -tuples of points $(x_1, \dots, x_g) \in \Sigma \times \dots \times \Sigma$ such that there exists a permutation $\sigma \in S_g$ and $x_i \in \alpha_i \cap \beta_{\sigma(i)}$.

Problem 12. Show that if each of the α -curves is transverse to each of the β -curves, then also \mathbb{T}_α intersects \mathbb{T}_β transversally.

Problem 13. Let T_α be the image of $H_1(\mathbb{T}_\alpha; \mathbb{Z})$ in $H_1(\text{Sym}^g(\Sigma); \mathbb{Z})$, let also T_β be the image of $H_1(\mathbb{T}_\beta; \mathbb{Z})$ in $H_1(\text{Sym}^g(\Sigma); \mathbb{Z})$. Prove that

$$H_1(\text{Sym}^g(\Sigma); \mathbb{Z}) / (T_\alpha + T_\beta) \cong H_1(\Sigma; \mathbb{Z}) / ([\alpha_1], \dots, [\beta_g]) \cong H_1(Y; \mathbb{Z}).$$

Chose two paths a and b , one belonging to \mathbb{T}_α , the other belonging to \mathbb{T}_β . Assume that they have the same endpoints $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. These two paths form a loop $\gamma \in \pi_1(\text{Sym}^g(\Sigma))$.

Problem 14. Prove that γ depends only on x and y and not on a and b .

Taking the solution of Problem 14 for granted, with each pair of points $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ we associate an element $\epsilon(x, y) \in H_1(Y; \mathbb{Z})$.

Problem 15. Prove that ϵ is additive in the sense that $\epsilon(x, y) + \epsilon(y, z) = \epsilon(x, z) \in H_1(Y; \mathbb{Z})$.

There exists another description of the class $\epsilon(x, y)$ (the reader might want to look back to Section 2.1 before reading this paragraph). To begin with, choose a point $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Each such point, by Problem 11, corresponds to a set of g -points x_1, \dots, x_g , such that $x_i \in \alpha_i \cap \beta_{\sigma(i)}$, where σ is some permutation of the set $\{1, \dots, g\}$. Each of the x_i corresponds to a trajectory γ_i of the vector field ∇F , which connects a critical point of index 1 to a critical point of index 2. There is also a unique trajectory γ_z passing through the point z . It connects the critical point of index 0 with the critical point of index 3. Take now small neighborhoods U_1, \dots, U_g, U_z of the trajectories $\gamma_1, \dots, \gamma_g, \gamma_z$. Let Y° be the complement $Y \setminus (U_1 \cup \dots \cup U_g \cup U_z)$. The vector field ∇F does not vanish on Y° . The pair $(Y^\circ, \nabla F)$ defines then a so-called smooth Euler structure on Y ; see [109]. By the result of Turaev, a smooth Euler structure corresponds to a Spin^c structure on Y [109, Proposition 2.7]. Call this structure \mathfrak{s}_x . Each Spin^c structure has its Chern class $c_1 \in H^2(Y; \mathbb{Z})$, as was discussed in Section 2.1.

Proposition 2.22 (see [82, Lemma 2.19]). *Given any two points $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, the difference $c_1(\mathfrak{s}_x) - c_1(\mathfrak{s}_y)$ is the Poincaré dual to $\epsilon(x, y)$.*

2.4. The chain complex \widehat{CF}

We will work mostly over \mathbb{Z}_2 . For simplicity, unless specified otherwise, we will assume that $b_1(Y) = 0$.

Let $(\Sigma, \alpha, \beta, z)$ be a pointed Heegaard diagram for Y . Assume that the α and β curves intersect transversally. Then, the chain complex \widehat{CF} is defined (over \mathbb{Z}_2) to be generated by the intersection points $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

Remark 2.23. There are a few technical assumptions on the Heegaard diagram used in the construction of the chain complex. First of all, we usually assume that $g > 2$ (case $g = 1$ is very special and also possible, see [82, Remark 2.16]); see Remark 2.20 for the case $g = 2$.

If $b_1(Y) > 0$, one adds an extra assumption on the Heegaard diagram, namely admissibility, see [82, Section 5]. For example, this condition rules out a diagram for $S^2 \times S^1$, where Σ is a torus and the α -curve and the β -curve are parallel, so $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is empty. An admissible Heegaard diagram for $S^2 \times S^1$ can be obtained by moving the β -curve by an isotopy in such a way that two intersection points with the α -curve are created.

We now define the differential ∂ . Let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ be two intersection points. Denote by $\pi_2(\mathbf{x}, \mathbf{y})$ the set of relative homotopy classes of disks $\phi: D^2 \rightarrow \text{Sym}^g(\Sigma)$ with $\phi(-1) = \mathbf{x}$, $\phi(1) = \mathbf{y}$, $\phi(\partial_+ D^2) \subset \mathbb{T}_\alpha$ and $\phi(\partial_- D^2) \subset \mathbb{T}_\beta$. Here D^2 is the unit disk in the complex plane, $\partial_\pm D^2$ is the part of the boundary having positive (respectively: negative) imaginary part.

Problem 16. Show that $\pi_2(\mathbf{x}, \mathbf{y})$ can be non-empty only if $\epsilon(\mathbf{x}, \mathbf{y}) = 0$.

Problem 17. Show that $\pi_2(\mathbf{x}, \mathbf{y})$ admits a multiplication defined as

$$\pi_2(\mathbf{x}, \mathbf{y}) \star \pi_2(\mathbf{y}, \mathbf{z}) \rightarrow \pi_2(\mathbf{x}, \mathbf{z})$$

Show that \star is associative. Prove also that $\pi_2(\mathbf{x}, \mathbf{x})$ is a group.

Problem 18. Show that there is an action of $\pi_2(\text{Sym}^g(\Sigma)) = \mathbb{Z}$ on each of the sets $\pi_2(\mathbf{x}, \mathbf{y})$.

Problem 19. Draw a standard $g = 1$ Heegaard diagram for a lens space $L(p, q)$. Show that there are precisely p intersection points of the α -curves with β -curves and $\epsilon(\mathbf{x}, \mathbf{y}) \neq 0$ as long as $\mathbf{x} \neq \mathbf{y}$.

Problem 20. Write explicitly all holomorphic maps from D^2 to D^2 that fix -1 and 1 and take $\partial_+ D^2$ to $\partial_+ D^2$. Show that the space of these maps can be parametrized by \mathbb{R} .

Given $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, a *holomorphic* representative for ϕ is a map $u: D^2 \rightarrow \text{Sym}^g(\Sigma)$ in the homotopy class ϕ that is holomorphic. Recall that $\text{Sym}^g(\Sigma)$ has a complex structure induced from Σ and D^2 has a standard complex structure.

Remark 2.24. For various genericity results, the complex structure on $\text{Sym}^g(\Sigma)$ induced from a complex structure on Σ might be too rigid and one often needs to consider almost complex structures (that is, endomorphisms of the tangent bundle whose square is minus the identity) and pseudo-holomorphic maps instead of holomorphic. We refer to [82, Section 3.1] for more details.

We denote by $\mathcal{M}(\phi)$ the space of holomorphic representatives of ϕ . For any class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, there is an integer $\mu(\phi) \in \mathbb{Z}$ called the *Maslov index*. A detailed definition of the Maslov index in Heegaard Floer theory can be found in [46, Section 4]. The Maslov index is the dimension of the moduli space of holomorphic representatives (provided the almost complex structure is sufficiently generic). By Problem 20, there is an action of \mathbb{R} on $\mathcal{M}(\phi)$ given by the automorphisms of the domain D^2 that fix 1 and -1 . If ϕ is not the class of a constant map, and the complex structure on Σ was generic, then the quotient $\widehat{\mathcal{M}}(\phi) = \mathcal{M}(\phi)/\mathbb{R}$ is a smooth manifold of dimension $\mu(\phi) - 1$; see [102]. If additionally $\mu(\phi) = 1$, we define

$$\#\widehat{\mathcal{M}}(\phi) \in \mathbb{Z}$$

to be the number of the elements in the quotient.

Remark 2.25. In [82, Section 3.6] there is described a way to associate a sign to each element $\widehat{\mathcal{M}}(\phi)$ as long as $\mu(\phi) = 1$. This allows us to define the differential in the Heegaard Floer theory over \mathbb{Z} . As we already mentioned above, we will mostly focus on the theory over \mathbb{Z}_2 .

The basepoint z can be used to construct a codimension two submanifold (in the language of algebraic geometry: a divisor), $R_z := \Sigma \times \dots \times \Sigma \times \{z\} \subset \text{Sym}^g(\Sigma)$ (the product is formally defined in $\Sigma^{\times g}$, we project it to $\text{Sym}^g(\Sigma)$).

Problem 21. Observe that, by construction, \mathbb{T}_α and \mathbb{T}_β are disjoint from R_Z .

Given intersection points $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and a class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, we define $n_Z(\phi)$ to be the intersection number between ϕ and R_Z .

The differential for \widehat{CF} is then given by

$$(2.2) \quad \partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ n_Z(\phi)=0, \mu(\phi)=1}} \# \widehat{\mathcal{M}}(\phi) \mathbf{y}.$$

In a few words, the differential counts holomorphic disks between \mathbf{x} and \mathbf{y} which do not intersect the divisor R_Z .

Problem 22. Show that for a lens space with a standard Heegaard diagram and $g = 1$, $\partial \mathbf{x} = 0$ for all $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

Problem 23. Take the standard diagram for $S^2 \times S^1$ with $g = 1$. Move the α -curve so that it intersects the β -curve at precisely two points \mathbf{x} and \mathbf{y} . Calculate the differential and the homology groups (compare Remark 2.23).

The following fact holds.

Theorem 2.26 (see [82, Theorem 4.1]). *We have $\partial^2 = 0$. The homologies $\widehat{HF}(Y)$ are independent of the choice of the Heegaard diagram, and, therefore, are invariants of the three-manifold Y .*

Remark 2.27. The words ‘independent of the choice’ might have different meanings. Originally, in [82], it was proved that a change of the Heegaard diagrams as in Section 2.2 above changes $\widehat{HF}(Y)$ by an isomorphism. Therefore, $\widehat{HF}(Y)$ was well defined up to isomorphism. In [35] Juhász and Thurston showed more, namely the naturality of the Heegaard Floer theory. Naturality means that the Heegaard Floer theory assigns a concrete group to each based¹ three-dimensional manifold and each diffeomorphism of a based manifold induces an isomorphism of corresponding Heegaard Floer groups. This naturality property is proved for all flavors of the Heegaard Floer homology. It lies at the heart of the involutive Floer theory as defined in [27] via the maps studied in detail in [104, 115]; see also [28].

Problem 24. Prove that $\widehat{HF}(Y)$ splits as a direct sum $\widehat{HF}(Y, \mathfrak{s})$ over all the Spin^c structures of Y .

Problem 25. Prove that if Y is a rational homology sphere, then $\widehat{HF}(Y, \mathfrak{s})$ is non-trivial for any Spin^c structure. In particular, $\text{rank} \widehat{HF}(Y) \geq |H_1(Y; \mathbb{Z})|$, where $|\cdot|$ denotes the cardinality of a set.

Definition 2.28 (L-space). A rational homology sphere is called an *L-space* if

$$\text{rank} \widehat{HF}(Y) = |H_1(Y; \mathbb{Z})|.$$

Problem 26. Prove that all the lens spaces are L-spaces.

2.5. Complexes CF^- , CF^+ and CF^∞

The complex structure on $\text{Sym}^g(\Sigma)$ and the holomorphicity of the maps used in the definition of \mathcal{M} were used to give rigidity to the space \mathcal{M} (to make sure it has a finite dimension).

The existence of this structure has one more consequence. Namely that the $n_Z(\phi)$ defined above is always non-negative. We will define a new chain complex, where we count all the holomorphic disks with $\mu(\phi) = 1$, regardless of the value of $n_Z(\phi)$. The chain complex CF^- is generated by the intersection points $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, but this time not over \mathbb{Z}_2 , but over the ring $\mathbb{Z}_2[U]$, where U is a formal variable. The differential for CF^- is defined by

¹A *based manifold* is a manifold with a choice of a base point.

$$(2.3) \quad \partial \mathbf{x} := \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi)=1}} \# \widehat{\mathcal{M}}(\phi) U^{n_z(\phi)} \mathbf{y}.$$

Theorem 2.29. *We have $\partial^2 = 0$. The homology groups $HF^-(Y)$ do not depend on the choice of the Heegaard diagram.*

Remark 2.27, explaining the meaning of ‘do not depend’, still applies in the case of HF^- . As before, the group $HF^-(Y)$ splits as a sum over the Spin^c structures of Y . We also have the following fact, which is not very hard to prove.

Proposition 2.30. *A three-manifold Y is an L-space if and only if, for every \mathfrak{s} , $HF^-(Y, \mathfrak{s}) \cong \mathbb{Z}_2[U]$.*

In algebra there is a procedure called localization, which roughly means inverting formally some variables in a ring. For example, the localization of $\mathbb{Z}_2[U]$ with respect to the multiplicative system generated by U is the ring $\mathbb{Z}_2[U, U^{-1}]$. We can perform this operation on the module CF^- : define a chain complex as generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, but this time over $\mathbb{Z}_2[U, U^{-1}]$. The chain complex will be denoted by CF^∞ . The differential is defined as in (2.3). The homology of the complex is well-defined and will be denoted by $HF^\infty(Y, \mathfrak{s})$. As it might be expected, by passing to a localization, we lose some information. Actually we lose a lot: namely, the following holds.

Theorem 2.31 (see [83, Theorem 10.1]). *Suppose Y is a rational homology sphere. We have an isomorphism of $\mathbb{Z}_2[U, U^{-1}]$ -modules*

$$HF^\infty(Y, \mathfrak{s}) \cong \mathbb{Z}_2[U, U^{-1}].$$

Remark 2.32. Theorem 2.31 allows generalizations for non rational homology spheres; again, see [83, Theorem 10.1].

The chain complex CF^- can be regarded as a subcomplex of CF^∞ . For this, we need to regard CF^∞ as a complex over $\mathbb{Z}_2[U]$. The quotient complex $CF^+(Y)$ is well defined. This is a chain complex over $\mathbb{Z}_2[U]$. The homologies are called $HF^+(Y)$.

Problem 27. Prove that for every element $a \in CF^+$ there exists $k \geq 0$ such that $U^k a = 0$.

The short exact sequence

$$0 \rightarrow CF^- \rightarrow CF^\infty \rightarrow CF^+ \rightarrow 0$$

gives rise to an exact triangle in homology.

Proposition 2.33. *There exists yet another short exact sequence*

$$0 \rightarrow \widehat{CF} \rightarrow CF^+ \xrightarrow{U} CF^+ \rightarrow 0$$

giving rise to a long exact sequence in homology.

Problem 28. Write precisely the two long exact sequences mentioned above. Watch out for grading shifts; these will be introduced below.

Problem 29. Prove that $HF^+(Y, \mathfrak{s})$ splits non-canonically as a sum of a part isomorphic to $\mathbb{Z}_2[U, U^{-1}]/(U)$ and a part finitely generated over \mathbb{Z}_2 . Show that Y is an L-space if and only if for every \mathfrak{s} we have $HF^+(Y, \mathfrak{s}) = \mathbb{Z}_2[U, U^{-1}]/(U)$ as $\mathbb{Z}_2[U]$ modules.

So far we have defined various chain complexes, but we have not defined a grading yet. We have the following useful Lemma.

Lemma 2.34 (see [82, Lemma 3.3], [90]). *If $g > 2$, then for any $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ the difference $\mu(\phi) - 2n_z(\phi)$ does not depend on the specific choice of ϕ , only on \mathbf{x} and \mathbf{y} .*

Lemma 2.34 allows us to define the relative grading of chain complexes. Namely, we define the *Maslov grading* $M(\mathbf{x}) - M(\mathbf{y}) = \mu(\phi) - 2n_z(\phi)$. The differential decreases the Maslov grading by 1, provided we require that the multiplication by U shifts the Maslov grading by -2 . Later on we will show that the Maslov grading gives rise to an absolute grading.

Problem 30. Suppose that (M_1, s_1) and (M_2, s_2) are two three-manifolds. Prove the following Künneth formula for \widehat{CF} :

$$\widehat{CF}(M_1 \# M_2, s_1 \# s_2) \cong \widehat{CF}(M_1, s_1) \otimes \widehat{CF}(M_2, s_2)$$

3. Why do things work?

It is not that hard to define invariants of three-manifolds. It is hard, though, to construct *meaningful* invariants. This means, invariants over which we have some control, and for which we can calculate some non-trivial estimates. In this section we are going to give two highly non-trivial results, which lie at the heart of the Heegaard Floer theory. These are the adjunction inequality and the surgery exact sequence. Many crucial results in Heegaard Floer theory rely on these two results.

3.1. Adjunction inequality

In algebraic geometry one has the so-called adjunction formula. In short if D is a smooth divisor in a projective variety X and K_D, K_X denote canonical divisors, then $K_D = (K_X + D)|_D$. For readers not acquainted with the language of algebraic geometry, one can think of K_D and K_X as (first Chern classes of) complex line bundles $K_D = \Lambda^{\dim D} T^*D$, $K_X = \Lambda^{\dim X} T^*X$ and the divisor D defines a complex line bundle, whose first Chern class is Poincaré dual to the class of D . The sum of divisors corresponds to a tensor product of line bundles and restriction means the restriction of line bundles in the ordinary sense. We refer to any textbook in algebraic geometry, like [24], for more details. With this setting, the adjunction formula is almost a tautology.

As a special case, suppose that C is a smooth complex curve in a projective surface X and K is the canonical divisor. We have that $K_C = (K_X + C)|_C$ and applying the classical Riemann–Roch theorem yields

$$(3.1) \quad \chi(C) = -C(C + K_X).$$

For example, if $X = \mathbb{C}P^2$ and C is a smooth complex curve of degree d , then in $H_2(X; \mathbb{Z})$ we have $C = dH$, $K = -3H$, where H is the class of a line and so $\chi(C) = -d(d - 3)$. Equation (3.1) is sometimes referred to as the adjunction equality.

It is trivial to see that the adjunction equality (3.1) has no chances to hold in a smooth category. For example, draw a genus g surface in \mathbb{C}^2 , it is a homologically trivial surface in the compactification $\mathbb{C}P^2$, so (3.1) would imply that $2 - 2g = 0$.

A wonderful tool in Seiberg–Witten theory is the *adjunction inequality*. Recall that Seiberg–Witten theory assigns to every Spin^c structure s on a smooth four-manifold X with $b_2^+(X) > 1$ an integer number $SW_X(s)$. We have the following remarkable theorem, which we state in a simple form, see e.g. [105, Section 10] for a more detailed version. Other sources are [42, Section 40] and [75, Section 4.6].

Theorem 3.1 (Adjunction inequality in Seiberg–Witten theory). *Suppose X is a smooth four-manifold with $b_2^+(X) > 1$. Let $C \subset X$ be a smooth closed connected embedded surface such that $C^2 \geq 0$ and C is homologically non-trivial. If s is a Spin^c structure on X such that $SW_X(s) \neq 0$, then $\chi(C) + C^2 \leq -|(c_1(s), C)|$.*

The assumption that C is smooth is essential. For example, in [44] there are constructed locally flat embedded surfaces C in $\mathbb{C}P^2$ such that $\chi(C) > -d(d - 3)$, where d is the degree of

C. This problem is related to showing that the topological four-genus of some algebraic knots is strictly less than the smooth four-genus; see [103, 1].²

In Heegaard Floer theory, the adjunction inequality is a key tool in proving many important theorems. The formulation below involves manifolds with $b_1 > 0$. In that case, the homology $HF^+(Y, \mathfrak{s})$ can be zero for some Spin^c structures, unlike in the case $b_1 = 0$ (cf. Problem 25).

Theorem 3.2 (Adjunction Inequality). *Suppose Y is a three-manifold with $b_1(Y) > 0$. Let \mathfrak{s} be a Spin^c structure for which $HF^+(Y, \mathfrak{s})$ is non-zero. Suppose $Z \subset Y$ is a smooth closed oriented surface and $g(Z) > 0$. Then $|\langle c_1(\mathfrak{s}), [Z] \rangle| \leq 2g(Z) - 2$.*

The adjunction inequality is proved in [83, Section 7].

3.2. The surgery exact sequence

One of the most important basic tools for calculating the Heegaard Floer invariants is the surgery exact sequence. The most basic form of it is often used as a template for proving more general statements. A surgery exact sequence exists in the Seiberg–Witten Floer theory (see for example [42, Section 42] and references therein). In Heegaard Floer theory, we have a way of calculating any surgery on a null-homologous knot in an integer homology three-sphere, provided we know its knot Floer chain complex; see [91] for details. This general surgery formula relies on the following fundamental result, see [83, Theorem 1.7].

Theorem 3.3 (Surgery Exact Sequence). *Let Y be an integral homology three-sphere and $K \subset Y$ be a knot. Then there exists a U -equivariant exact sequence:*

$$\dots \rightarrow HF^+(Y) \rightarrow HF^+(Y_0) \rightarrow HF^+(Y_1) \rightarrow HF^+(Y) \rightarrow \dots$$

where Y_1 is the $+1$ surgery and Y_0 is the 0 surgery on K .

The surgery exact sequence is proved in [83, Section 9]. The key idea is to find a triple Heegaard diagram, that is a quintuple $(\Sigma, \alpha, \beta, \gamma, z)$, such that $(\Sigma, \alpha, \beta, z)$ is a Heegaard diagram for Y , $(\Sigma, \alpha, \gamma, z)$ is a Heegaard diagram for Y_0 and $(\Sigma, \beta, \gamma, z)$ is a Heegaard diagram for Y_1 . The details and the proof of the existence of such a triple Heegaard diagram are given in [83, Lemma 9.2]. Speaking very roughly, given the Heegaard diagram, the maps in the surgery long exact sequence are built by counting holomorphic triangles, instead of holomorphic disks.

4. Cobordisms and d -invariants.

4.1. Absolute grading

This section is based on [79].

Definition 4.1. Let $(Y_1, \mathfrak{s}_1), (Y_2, \mathfrak{s}_2)$ be two Spin^c three-manifolds. We say that (W, \mathfrak{t}) is a Spin^c cobordism between Y_1 and Y_2 if W is a smooth four-manifold with boundary $Y_2 \sqcup -Y_1$ and \mathfrak{t} is a Spin^c structure on W whose restriction to Y_i is \mathfrak{s}_i , $i = 1, 2$.

Theorem 4.2 (see e.g. [79, Section 2]). *If (W, \mathfrak{t}) is a smooth Spin^c cobordism between (Y_1, \mathfrak{s}_1) and (Y_2, \mathfrak{s}_2) , then there exist maps $F_{W, \mathfrak{t}}^\bullet : HF^\bullet(Y_1, \mathfrak{s}_1) \rightarrow HF^\bullet(Y_2, \mathfrak{s}_2)$ with $\bullet \in \{+, -, \infty\}$, making the following diagram commute*

$$(4.1) \quad \begin{array}{ccccccc} \dots & \longrightarrow & HF^-(Y_1, \mathfrak{s}_1) & \longrightarrow & HF^\infty(Y_1, \mathfrak{s}_1) & \longrightarrow & HF^+(Y_1, \mathfrak{s}_1) \longrightarrow \dots \\ & & \downarrow F_{W, \mathfrak{t}}^- & & \downarrow F_{W, \mathfrak{t}}^\infty & & \downarrow F_{W, \mathfrak{t}}^+ \\ \dots & \longrightarrow & HF^-(Y_2, \mathfrak{s}_2) & \longrightarrow & HF^\infty(Y_2, \mathfrak{s}_2) & \longrightarrow & HF^+(Y_2, \mathfrak{s}_2) \longrightarrow \dots \end{array}$$

²Of course, one can complain that $b_2^+(CP^2) = 1$, so technically speaking locally flat curves in CP^2 are not counterexamples to the statement of Theorem 3.1, but they give an idea of the reason why Theorem 3.1 does not hold in the topological locally flat category.

The idea of the proof is to split the cobordism into handle attachments. The non-trivial part comes from two-handle attachments, which are basically dealt with using a refined version of the surgery exact sequence. We define a relative grading of the map induced by F .

Theorem 4.3 (see [88, Theorem 7.1]). *The map $F_{W,t}^\bullet$ has relative Maslov grading equal to*

$$\deg F_{W,t}^\bullet := \frac{c_1(t)^2 - 2\chi(W) - 3\sigma(W)}{4}.$$

We can now make the gradings in Heegaard Floer homology groups absolute by requiring that the generator of $HF^-(S^3)$ be at Maslov grading -2 , or, equivalently, that the lowest grading of $HF^+(S^3)$ be at Maslov grading 0 .

4.2. The d -invariants

The fact that $F_{W,t}$ preserves the grading is very interesting, but on its own does not give much of insight in the behavior of Heegaard Floer homology under cobordisms. A reader with some experience in Khovanov homology surely knows that the map in Khovanov homology induced by a knot cobordism has a fixed grading, but we do not know much more about this map; even the question whether it is non-trivial is not well understood.

Luckily, in the Heegaard Floer case, we have the following crucial fact.

Theorem 4.4 (see [79, Proof of Theorem 9.1]). *If W has negative definite intersection form, and Y_1, Y_2 are rational homology spheres, then $F_{W,t}^\infty$ is an isomorphism. On the contrary, if $b_2^+(W) > 0$, then $F_{W,t}^\infty$ is the zero map.*

Definition 4.5. Let (Y, \mathfrak{s}) be a rational homology three-sphere. The d -invariant or the correction term $d(Y, \mathfrak{s})$ is defined as the minimal absolute grading of a non-trivial element $x \in HF^+(Y, \mathfrak{s})$ which is in the image of $HF^\infty(Y, \mathfrak{s})$.

Let (W, \mathfrak{t}) be a Spin^c cobordism between (Y_1, \mathfrak{s}_1) and (Y_2, \mathfrak{s}_2) . The main result related to the d -invariants is the following.

Theorem 4.6 (see [79, proof of Theorem 9.9]). *Suppose (W, \mathfrak{t}) is a Spin^c cobordism between rational homology spheres (Y_1, \mathfrak{s}_1) and (Y_2, \mathfrak{s}_2) . If $b_2^+(W) = 0$, then*

$$(4.2) \quad d(Y_2, \mathfrak{s}_2) - d(Y_1, \mathfrak{s}_1) \geq \frac{1}{4}(c_1(t)^2 - 2\chi(W) - 3\sigma(W)).$$

Problem 31. Using (4.1) and Theorem 4.4, prove Theorem 4.6.

The d -invariants are strong enough to prove Donaldson's diagonalization theorem via Elkies' theorem; see [79, Section 9]. A version of d -invariants for manifolds with $b_1 > 0$, whose rudiments were established in [79], and which was developed in full details in [45], can be used to reprove the Kronheimer-Mrowka result on the smooth four-genus of torus knots. We refer again to [79, Section 9].

We gather now a few facts about the d -invariant, the first one is proved in [79, Theorem 4.3], while the second is proved in [79, Proposition 4.2].

Proposition 4.7.

- *The d -invariant is additive. That is, if (Y_1, \mathfrak{s}_1) and (Y_2, \mathfrak{s}_2) are two rational homology three-spheres, then $d(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2) = d(Y_1, \mathfrak{s}_1) + d(Y_2, \mathfrak{s}_2)$.*
- *Let (Y, \mathfrak{s}) be a rational homology three-sphere. Then $d(-Y, \mathfrak{s}) = -d(Y, \mathfrak{s})$.*

The first part of the proposition follows essentially from the Künneth principle (with some technical problems in homological algebra). However, the second part is more difficult than one could expect.

Using second part of Proposition 4.7 together with Theorem 4.6 we obtain the following result.

Corollary 4.8. *If (Y, \mathfrak{s}) bounds a rational homology ball W (that is, if $H_k(W; \mathbb{Q}) = 0$ for $k \geq 1$) and the Spin^c structure \mathfrak{s} extends over W , then $d(Y, \mathfrak{s}) = 0$.*

Problem 32. Prove Corollary 4.8.

We will be able to calculate the d -invariants for a large class of three-manifolds using Heegaard Floer homology for knots. This theory, usually called knot Floer theory, will be discussed in the next section.

Problem 33. Drill two balls from $\mathbb{C}P^2$ so as to obtain a cobordism between two copies of S^3 . Find all Spin^c structures on the cobordism that extend the Spin^c structure on S^3 (use Corollary 2.9). Use this example to show that Theorem 4.6 dramatically fails if $b_2^+(W) > 0$.

5. Heegaard Floer homology for knots

There is a variant of Heegaard Floer homology for knots and links. We will focus on knots in S^3 , although a significant part of the results carries through to null-homologous knots in rational homology spheres. The case of links, though, does not seem to be more complicated at the beginning, but there are surprisingly many highly non-trivial technical problems, e.g. if one tries to establish a surgery formula. The reader with some experience in link theory might think that Heegaard Floer homology for links is more complicated than for knots in a similar manner as Blanchfield forms for links are way more complicated than for knots.

5.1. Heegaard diagrams and knots

Suppose Y is a three-manifold and $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a Heegaard diagram for Y . Choose two base points z and w in $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$. Such quintuple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ is called a *doubly pointed Heegaard diagram*.

Given a doubly pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ we not only recover the manifold Y , but we obtain a way to encode a knot in Y . To this end, suppose the Heegaard decomposition is $Y = U_0 \cup_{\Sigma} U_1$. Connect points w and z by two curves $a \subset \Sigma \setminus \{\alpha_1, \dots, \alpha_g\}$, $b \subset \Sigma \setminus \{\beta_1, \dots, \beta_g\}$, and then push a into U_0 and b into U_1 . These two curves together result in a knot $K \subset Y$.

Problem 34. Prove that the isotopy type of K does not depend on the actual choice of the curves a and b .

Conversely, a knot $K \subset Y$ determines a doubly pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$. We focus on the case $Y = S^3$. Take a *bridge presentation* of K , i.e., its projection with a division of K into $2g + 2$ segments (for some $g \geq 0$) $a_1, \dots, a_{g+1}, b_1, \dots, b_{g+1} \subset K$, such that all the crossings are only between segments a_i and b_j and in such a way that, for every intersection, a_i always goes transversely over b_j (see Figure 5.1).

Consider the plane with this projection as $\{z = 0\} \subset \mathbb{R}^3$ and add to it a point at infinity, so that we may consider it as a subset of a 2-sphere $S^2 \subset S^3$. Let us define β_1, \dots, β_g as boundaries of some small pairwise non-intersecting tubular neighborhoods of b_1, \dots, b_g in this sphere. Now attach to the resulting sphere $g + 1$ handles at the endpoints of segments a_1, \dots, a_{g+1} in such a way that β_1, \dots, β_g encircle attaching discs of handles a_1, \dots, a_g respectively. We imagine these handles as sitting above the plane, i.e., as subsets of $\{z \geq 0\} \subset \mathbb{R}^3$. By this construction we clearly obtain a genus $g + 1$ surface Σ . We define the remaining β_{g+1} curve as a meridian of the handle corresponding to the curve a_{g+1} . Finally, define the loops $\alpha_1, \dots, \alpha_{g+1}$ as curves going along these attached handles and connected at the ends via the remaining parts of a_1, \dots, a_{g+1} , respectively. We arrange all the intersections to be transversal. This is the *stabilized Heegaard diagram* $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ associated to the knot K ; see Figure 5.2.

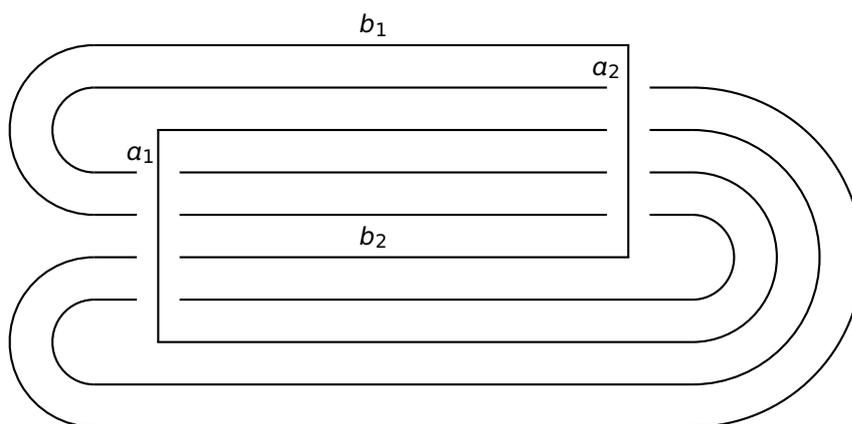


Figure 5.1: A bridge presentation of a figure-eight knot.

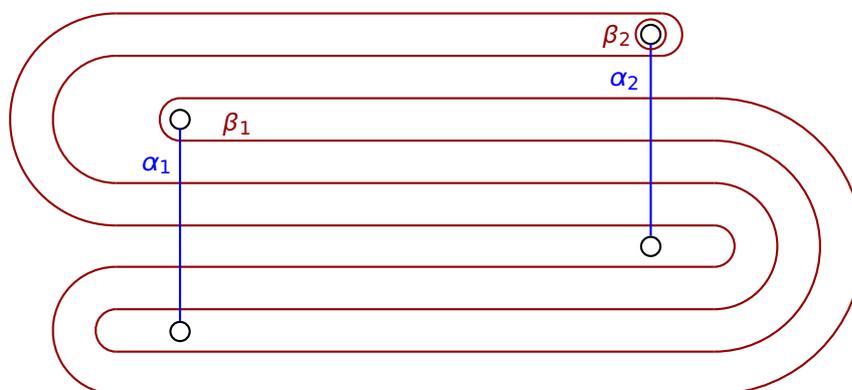


Figure 5.2: A stabilized diagram (Σ, α, β) associated to a figure-eight knot bridge presentation from Figure 5.1. Empty circles at the endpoints of α_i denote the disks where the handles are attached.

Problem 35. Show that the stabilized Heegaard diagram (Σ, α, β) constructed above represents S^3 .

For the construction of a chain complex associated to a knot K we need to introduce base-points. They are obtained by destabilizing the diagram (Σ, α, β) (cf. Theorem 2.17). Namely, we forget about the curves $\alpha_{g+1}, \beta_{g+1}$, and remove the handle associated to the curve α_{g+1} , defining points w, z as the endpoints of α_{g+1} . This results in a *destabilized Heegaard diagram* $(\Sigma, \alpha, \beta, w, z)$, where Σ is now a surface of a genus g , and $\alpha = \{\alpha_1, \dots, \alpha_g\}$, $\beta = \{\beta_1, \dots, \beta_g\}$; see Figure 5.3.

Problem 36. At the beginning of Section 5.1 we described a recipe for obtaining a knot from a doubly pointed Heegaard diagram and later we sketched a way to obtain a doubly pointed Heegaard diagram from a knot. Show that if one starts with an arbitrary knot $K \subset S^3$, passes to a Heegaard diagram and then recovers a knot K' from the Heegaard diagram, then K' is isotopic to K .

Remark 5.1. For simplicity we described a construction of a doubly pointed Heegaard diagram from a knot in S^3 . We refer to [84, Section 2.2] for a construction of Heegaard diagrams for a knot in an arbitrary three-manifold.



Figure 5.3: A destabilized version of the Heegaard diagram 5.2, with the intersection points $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ depicted.

5.2. The hat chain complex associated to a doubly pointed Heegaard diagram

Consider a doubly pointed Heegaard diagram $(\Sigma, \alpha, \beta, z, w)$ representing (Y, K) . Let $g = g(\Sigma)$. We define real g -dimensional tori $\mathbb{T}_\alpha, \mathbb{T}_\beta \subset \text{Sym}^g(\Sigma)$ as in Section 2.3 above. Moreover, let $R_z, R_w \subset \text{Sym}^g(\Sigma)$ be given by $(\{z\} \times \Sigma \times \dots \times \Sigma)/S_g$ and $(\{w\} \times \Sigma \times \dots \times \Sigma)/S_g$.

The chain complex $\widehat{CFK}(Y, K)$ is generated by the intersection points $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. For any pair $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ we define the *relative Maslov grading*

$$M(\mathbf{x}) - M(\mathbf{y}) = \mu(\phi) - 2n_w(\phi),$$

where $n_w(\phi)$ is the intersection index of ϕ and R_w . Likewise, we define the *relative Alexander grading*

$$(5.1) \quad A(\mathbf{x}) - A(\mathbf{y}) = n_z(\phi) - n_w(\phi).$$

Various aspects on the Alexander grading are elaborated in [100, Section 4]. If $Y = S^3$, there is a way of fixing the Maslov grading, so that it becomes an absolute grading (over \mathbb{Z}). We refer to [55, Section 3.4] for more details.

Proposition 5.2 (see [84, Section 1.1]). *If $Y = S^3$, then there exists a way of assigning the absolute Alexander grading $A(\mathbf{x})$ in such a way that (5.1) holds and moreover*

$$\sum_{\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} (-1)^{M(\mathbf{x})} t^{A(\mathbf{x})} = \Delta_K(t),$$

where $\Delta_K(t)$ is the symmetrized Alexander polynomial of the knot K .

Now we come to a potential source of confusion, because there are two choices of a differential in $\widehat{CFK}(Y, K)$. We can either set:

$$\partial_{grad} \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ n_z(\phi) = n_w(\phi) = 0 \\ \mu(\phi) = 1}} \# \widehat{\mathcal{M}}(\phi) \mathbf{y},$$

or

$$\partial_{fil} \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ n_w(\phi) = 0 \\ \mu(\phi) = 1}} \# \widehat{\mathcal{M}}(\phi) \mathbf{y}.$$

Problem 37. Prove that ∂_{grad} preserves the Alexander grading, while ∂_{fil} is a filtered map with respect to the Alexander grading.

The map ∂_{fil} is the differential in the complex $\widehat{CF}(Y)$, hence $\partial_{fil}^2 = 0$ (by Theorem 2.26), and ∂_{grad} is a part of ∂_{fil} that preserves the Alexander grading, we have that $\partial_{grad}^2 = 0$; compare [100, Section 4.4].

Problem 38. Show also that the homology of $(\widehat{CFK}(Y, K), \partial_{fil})$ is isomorphic to $\widehat{HF}(Y)$.

Definition 5.3. The homology of the complex $(\widehat{CFK}(Y, K), \partial_{grad})$ is called the *hat knot Floer homology* and denoted $\widehat{HFK}(Y, K)$.

From the point of view of homological algebra, if we have a filtered complex, like in our case $(\widehat{CFK}(Y, K), \partial_{fil})$, we can associate with it a graded complex, whose underlying space is isomorphic (at least if the complex is defined over a field), and with the differential consisting only of the graded part. In our case this is $(\widehat{CFK}(Y, K), \partial_{grad})$. There is a spectral sequence whose first page is the homology of the graded part, which abuts (under some finiteness assumptions on the complex, which are satisfied in Heegaard Floer theory) to the homology of the filtered complex. This spectral sequence is used in [84] to define an important knot invariant, called the τ -invariant. We will not discuss it here.

Theorem 5.4 (see [84, Corollary 3.2], [100, Theorem 1]). *The homology $\widehat{HFK}(Y, K)$ is a knot invariant. Moreover,*

$$\sum_a (-1)^{M(a)} t^{A(a)} = \Delta_K(t),$$

where the sum is taken over a graded basis of $\widehat{HFK}(Y, K)$.

One of the consequences of the adjunction inequality is the following result; see [84, Theorem 5.1].

Theorem 5.5 (Adjunction inequality in $\widehat{HFK}(Y, K)$). *Suppose that $K \subset Y$ is a null-homologous knot. Suppose s is such that $\widehat{HFK}(Y, K, s) \neq 0$. Then for every Seifert surface F for K of genus $g > 0$ we have*

$$|\langle c_1(s), F \rangle| \leq 2g(F).$$

The knot Floer homologies have two wonderful properties. The first one was proved in [85], the second one is proved in [20, 72].

Theorem 5.6. *The following two properties hold:*

- *If K is a knot in S^3 , then \widehat{HFK} detects the three-genus. More precisely, for a knot $K \subset S^3$,*

$$g_3(K) = \max_a: \widehat{HFK}_*(K, a) \neq 0.$$

- *\widehat{HFK} detects fibredness. That is, for a null-homologous knot K in a closed, oriented, connected 3-manifold, K is fibered if and only if*

$$\text{rank } \widehat{HFK}_*(K, g_3(K)) = 1.$$

Here $\widehat{HFK}_*(K, a)$ denotes the part of \widehat{HFK} with the Alexander grading a .

Remark 5.7. The fact that \widehat{HFK} detects the three-genus of a knot, can be generalized for null-homologous knots in rational homology three-spheres, where the notion of the three-genus is replaced by the Thurston norm; see [85, Section 1] and [73]. The fibreness part works for arbitrary null-homologous knots in arbitrary closed three-manifold; see [72].

5.3. The complexes CFK^- and CFK^∞

The chain complex CFK^- is built in an analogous way, although some subtleties arise. The generators are again intersection points $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, the complex is defined over $\mathbb{Z}_2[U]$ and the Maslov and Alexander gradings are as above. The multiplication by U by definition decreases the Alexander grading by 1. The differential is the following

$$(5.2) \quad \partial \mathbf{x} := \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi)=1}} \# \widehat{\mathcal{M}}(\phi) U^{n_w(\phi)} \mathbf{y}.$$

The only difference with respect to (2.3) is that in the exponent of U we have n_w and not n_z . In the sense of Section 5.2, the differential should be called ∂_{fil} . If we take the graded differential, that is, the one that does not count discs crossing the first base point (that is, one adds the condition $n_z(\phi) = 0$ in the sum in (5.2)), we will get a graded chain complex. In [55, Section 3.4] this complex is denoted $gCFK^-$ and the homology is HFK^- .

Unlike in the hat version, we are not as much interested in the graded complex as in the filtered one, that is, the one with the differential given by (5.2). Even though the homologies of complexes $CFK^-(Y, K)$ and $CF^-(Y)$ are the same, there is a substantial difference between $CFK^-(Y, K)$ and $CF^-(Y)$. Namely, in $CFK^-(Y, K)$ we have the Alexander grading. The differential does not necessarily preserve the grading, but as multiplication by U drops the Alexander grading by 1, we will obtain that the differential never increases the grading.

Problem 39. Check that the last statement is true.

This means that $CFK^-(Y, K)$ is a filtered chain complex over $\mathbb{Z}_2[U]$, or a bifiltered chain complex over \mathbb{Z}_2 (with the other filtration given by powers of U , we will explain this in a while). This filtration is independent of the choice of the Heegaard diagram, in fact we have the following fact; see [84, 100].

Theorem 5.8. *The filtered chain homotopy type of $CFK^-(Y, K)$ is an invariant of the isotopy type of the knot.*

As it might be expected, the filtered chain homotopy type of $CFK^-(Y, K)$ contains much more information than just the homology of the chain complex. The famous saying of Andrew Ranicki, one of the inventors of algebraic surgery theory:

Motto (Ranicki). *“Chain complexes are good, homologies are bad”*

is very true also in Heegaard Floer theory.

As in Section 2.5 above, we can invert formally the variable U to obtain another chain complex, called CFK^∞ . Here we give a slightly different point of view of this object.

Consider a chain complex whose generators are triples $[\mathbf{x}, i, j]$ such that $i, j \in \mathbb{Z}$ and $A(\mathbf{x}) = j - i$. The triple $[\mathbf{x}, i, j]$ will correspond to the generator $U^{-i}\mathbf{x}$. The differential is as in (5.2).

Problem 40. Show that with this notation the definition in (5.2) boils down to

$$(5.3) \quad \partial[\mathbf{x}, i, j] = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi)=1}} \# \widehat{\mathcal{M}}(\phi)[\mathbf{y}, i - n_w(\phi), j - n_z(\phi)].$$

The chain complex with such a differential is denoted by $CFK^\infty(Y, K)$. The homology is clearly $HFK^\infty(Y, K) \cong HFK^\infty(Y)$. The chain complex admits an action of U , namely $U[\mathbf{x}, i, j] = [\mathbf{x}, i - 1, j - 1]$. One of the advantages of (5.3) over (5.2) is that the symmetry between the first and the second filtration levels is clearly seen in (5.3). This symmetry is a generalization of the symmetry of the Alexander polynomial of a knot.

Problem 41. Prove that the subcomplex $CFK^\infty(Y, K)\{i \leq 0\}$ is the chain complex $CFK^-(Y, K)$.

Remark 5.9. Sometimes one considers $CFK^- = CFK^\infty(Y, K)\{i < 0\}$, instead of $CFK^\infty(Y, K)\{i \leq 0\}$. This does not affect the isomorphism type of the relatively graded complex.

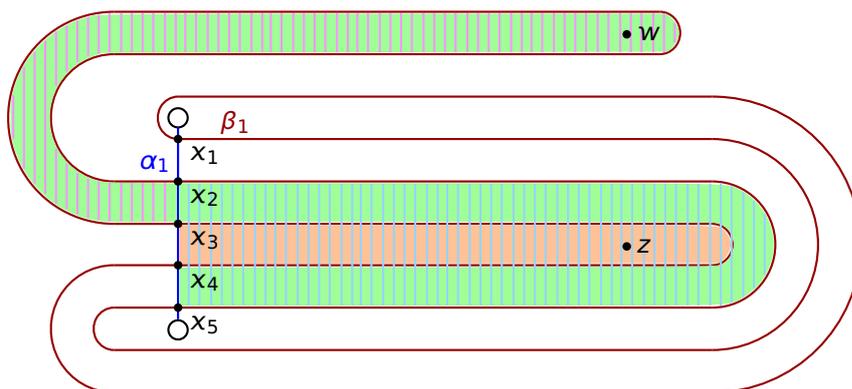


Figure 5.4: Holomorphic disks in a Heegaard diagram 5.3. Two disks at infinity that connect pairs (x_2, x_1) and (x_4, x_1) , and pass through points z and w respectively, are not shown.

The definition of CFK^∞ via $[\mathbf{x}, i, j]$ allows us to present it graphically. Namely, for any element $[\mathbf{x}, i, j]$ we can put a dot in a plane with coordinates (i, j) . The arrows denote differentials, often one draws only an edge, the direction of an arrow can be determined by the fact that the differential does not increase any of the two filtration levels. The Maslov grading is usually not presented, or denoted near the dots, if necessary.

One of the features of the chain complex CFK^∞ is its behavior under connected sums, which is an analogue of Problem 30.

Proposition 5.10. *Suppose K_1, K_2 are two knots in S^3 . Then*

$$CFK^\infty(K_1 \# K_2) \cong CFK^\infty(K_1) \otimes CFK^\infty(K_2)$$

where “ \cong ” denotes a bifiltered chain homotopy equivalence. The tensor product is taken over the ring $\mathbb{Z}_2[U, U^{-1}]$.

Problem 42. Take two knots K_1 and K_2 . Draw a knot diagram for K_1 and K_2 and connect them by a band to obtain a knot diagram for $K_1 \# K_2$; try to control the bridge presentation. Using Section 5.1 calculate $CFK^\infty(K_1 \# K_2)$ and prove as much as you can of Proposition 5.10 (existence of maps, gradings, filtrations, etc).

Example 5.11. Let us revisit the example of a figure-eight knot. The underlying surface of the Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ (see Figure 5.3) is of genus 1, thus its universal cover is \mathbb{C} . Therefore, by combining this fact with the Riemann mapping theorem, we get that if there exists a topological disk $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, then it is uniquely represented by a holomorphic disk. Using this fact it is straightforward to find all holomorphic disks as in a Figure 5.4.

From the same diagram we may find some of the relative Alexander gradings according to (5.1). This, together with Proposition 5.2, gives us a way to determine the absolute Alexander gradings $A(x_1), A(x_3), A(x_5) = 0, A(x_2) = 1, A(x_4) = -1$. We can also read off the differentials from Figure 5.4, according to Problem 40; the nontrivial ones are $\partial x_2 = x_1 + x_5, \partial x_4 = Ux_1 + Ux_5, \partial x_3 = Ux_2 + x_4$. In the chain complex CFK^∞ this means that $\partial[x_2, i, i + 1] = [x_1, i, i] + [x_5, i, i], \partial[x_4, i, i - 1] = [x_1, i - 1, i - 1] + [x_5, i - 1, i - 1], \partial[x_3, i, i] = [x_2, i - 1, i] + [x_4, i, i - 1]$ for $i \in \mathbb{Z}$. For convenience let us change variables, setting $x'_1 := x_1 + x_5$.

The complex $CFK^\infty(S^3, 4_1)$, spanned by the elements $[x'_1, i, i]$ and $[x_k, i, i + A(x_k)]$, where $k = 2, \dots, 5$, is depicted in Figure 5.5.

Example 5.12. Similarly, one can compute a complex $CFK^\infty(S^3, 3_1)$ and then use the Künneth formula (see Proposition 5.10) to obtain a full complex $CFK^\infty(S^3, 3_1 \# 3_1)$ of the connected sum of two copies of trefoils. Figure 5.6, after tensoring with $\mathbb{Z}_2[U, U^{-1}]$, presents the result after a change of basis.

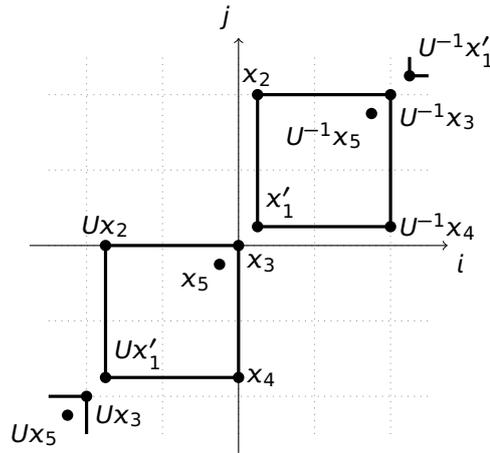


Figure 5.5: A complex representing $CFK^\infty(S^3, 4_1)$. Note that the elements $U^{-1}x'_1, U^{-1}x_3, U^{-1}x_5$ are in the same bifiltration level $(i, j) = (1, 1)$, likewise their images under the endomorphism U .

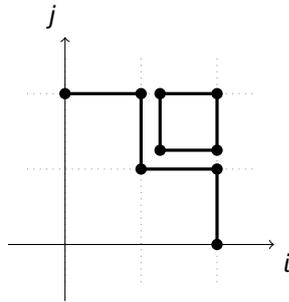


Figure 5.6: Tensoring this complex with $\mathbb{Z}_2[U, U^{-1}]$ results in the complex $CFK^\infty(S^3, 3_1 \# 3_1)$.

Problem 43. Calculate Example 5.12 by yourself.

Even though the homology of $CFK^\infty(Y, K)$ is not very interesting, the bifiltered chain homotopy type of the complex contains a lot of information about the knot. An important example of a piece of information contained in the chain complex $CFK^\infty(Y, K)$ that is lost when passing to homology is given below.

5.4. The V_m invariants

Let $K \subset S^3$ be a knot. For any $m \in \mathbb{Z}$ let $CFK^\infty(i < 0, j < m)$ be the subcomplex of CFK^∞ generated by elements at bifiltration level (i, j) , where $i < 0$ and $j < m$. Let A_m^+ be the quotient complex $CFK^\infty / CFK^\infty(i < 0, j < m)$.

Remark 5.13. Sometimes one writes that A_m^+ is a complex generated by elements at filtration level (i, j) , where $i \geq 0$ or $j \geq m$, and if a differential of an element leads out of A_m^+ we set it to be zero. This might be sometimes convenient but is not very rigorous, because it suggests that A_m^+ is a subcomplex of CFK^∞ , while it is not. If an element $x \in CFK^\infty$ is at filtration level $i \geq 0$ or $j \geq m$, and $\partial x = y$ with $y \in CFK^\infty(i < 0, j < m)$, then $\partial x = 0$ in A_m^+ by definition.

Definition 5.14. The V_m invariant of a knot K is minus one half of the minimal grading of a cycle $x \in A_m^+$, which is non-trivial in homology and such that for any $k \geq 0$ there exists $y_k \in A_m^+$ such that $U^k y_k = x$.

Remark 5.15. The notation V_m for these invariants is taken from [74]. In the original source, that is, Rasmussen's thesis [100], a related invariant h_k was studied.

Problem 44. Find a relation between V_m and the invariant h_k defined in Section 7.2 of the Rasmussen's thesis.

Problem 45. Notice that $V_m \leq V_{m-1}$. Prove that $V_{m-1} \leq V_m + 1$.

Problem 46. Calculate V_m for the sum of two trefoils and for the figure-eight knot. Observe that the 'squares' in both chain complexes do not contribute to V_m .

Proposition 5.16. *The number V_m is a concordance invariant.*

A proof using the large surgery formula is given as Problem 49. The original proof of [100] uses a different approach.

5.5. Large integer surgeries

There is a general way for calculating HF^+ (and so the d -invariants) of surgeries on a knot in S^3 , see for instance [91], once the chain complex $CFK^\infty(K)$ is known. Notice that knowing only HFK^- or \widehat{HFK} is usually not enough; recall Ranicki's motto. The general formula simplifies a lot, when the surgery coefficient is a large positive integer. Before we begin, we need to show a useful way of enumerating Spin^c structures on surgeries on a knot in S^3 . The following result can be found in [79, Lemma 7.10].

Proposition 5.17. *Let $q > 0$ be an integer and consider a knot $K \subset S^3$. Let $Y = S^3_q(K)$ and let W be a four-dimensional handlebody obtained by gluing a two-handle to the ball B^4 along a product neighborhood of K with framing q , so that $\partial W = Y$. Let $F \subset W$ be a closed surface obtained by capping a Seifert surface for K by the core of the two-handle.*

For any integer $m \in [-q/2, q/2)$ there exists a unique Spin^c structure s_m on Y characterized by the fact that it extends to a Spin^c structure t_m on W with the property that $\langle c_1(t_m), F \rangle + q = 2m$.

Problem 47.

- Prove that the definition of s_m does not depend on the choice of the Seifert surface used to construct F .
- Explain the action of $H^2(Y; \mathbb{Z})$ on the set of the Spin^c structures under the identification in Proposition 5.17.

Now we are ready to state the Large Surgery Theorem.

Theorem 5.18 (see [84, Theorem 4.4]). *Suppose that $K \subset S^3$ and $q \geq 2g_3(K) - 1$. Then for any Spin^c structure s_m (with $m \in [-q/2, q/2) \cap \mathbb{Z}$), we have an isomorphism between A_m^+ and $HF^+(S^3_q(K), s_m)$. The isomorphism changes the Maslov grading by $\frac{(q-2m)^2 - q}{4q}$. In particular, we have $d(S^3_q(K), s_m) = \frac{(q-2m)^2 - q}{4q} - 2V_m(K)$.*

As a corollary we give a proof of the concordance invariance of V_m . Suppose K is concordant to K' . Let $m \in \mathbb{Z}$ and choose a sufficiently large integer q , in particular we require that $q \geq \max\{2g_3(K) - 1, 2g_3(K') - 1, 2|m| + 1\}$. The d -invariants of q -surgery on K and K' are given by Theorem 5.18, therefore the invariance of V_m under a concordance follows from the following fact.

Lemma 5.19. *Suppose K is concordant to K' and $q > 0$. Then there exists a four-manifold W whose boundary is $S^3_q(K') \sqcup -S^3_q(K)$ and such that the inclusions $S^3_q(K) \hookrightarrow W$, $S^3_q(K') \hookrightarrow W$ induce isomorphisms on \mathbb{Z} homology. Moreover, for any integer $m \in [-q/2, q/2)$ there exists a Spin^c structure t_m on W extending the Spin^c structures s_m on both sides of the boundary.*

Problem 48. Consider the following construction. Let $A \subset S^3 \times [0, 1]$ be a concordance between K and K' . Glue a two-handle to $S^3 \times [0, 1]$ along a product neighborhood of $K' \subset S^3 \times \{1\}$ with framing q . Denote by W' the resulting four-manifold. Let $P \subset W'$ be the union of A and the core of the two handle and let N be a product neighborhood of P in W' . Show that $W = \overline{W' \setminus N}$ has all the properties stated in Lemma 5.19. See [4] for a generalization of this construction.

Problem 49. Conclude the proof of concordance invariance of V_m .

Problem 50. Let x_1, \dots, x_n be all the chains of $CFK^\infty(K)$, which are cycles and which are at grading 0. Prove that

$$V_m(K) = \min_{k=1, \dots, n} \max(i(x_k), j(x_k) - m),$$

where $i(x), j(x)$ denote the i -th and the j -th bifiltration levels as described in Section 5.3 above.

Problem 51. Show by means of an example, that V_m are in general not additive, that is, $V_m(K \# K')$ is not always equal to $V_m(K) + V_m(K')$.

Problem 52. Show that for all $k, m \in \mathbb{Z}$ $V_m(K \# K') \leq V_k(K) + V_{m-k}(K')$.

5.6. L-space knots

We will now introduce a class of knots for which the chain complex CFK^∞ is especially easy to describe.

Definition 5.20. A knot $K \subset S^3$ is called an *L-space knot* (sometimes called a positive L-space knot), if there exists a coefficient $q > 0$ such that $S_q^3(K)$ is an L-space.

The notion of an L-space knot was introduced in [86] in the context of the Berge conjecture, which predicts the list of all possible knots in S^3 such that a surgery on these knots with some coefficient gives a lens space. The notion of an L-space knot turns out to be very useful also for studying singularities of plane curves.

Example 5.21. By the result of Moser [65, Proposition 3.2], if $|pqr - s| = 1$, then the s/r -surgery on a positive torus knot $T(p, q)$ is the lens space $L(|s|, rq^2)$. Therefore, every positive torus knot is an L-space knot.

We have the following properties of L-space knots.

Lemma 5.22.

- (a) *L-space knots are prime. A connected sum of two non-trivial knots is never an L-space knot (see [40]).*
- (b) *If K is an L-space knot, then $S_q^3(K)$ is an L-space if and only if $q \geq 2g_3(K) - 1$ (see [92, Proposition 9.6] and [29]).*
- (c) *L-space knots are quasipositive (see [25]).*
- (d) *L-space knots are fibered.*
- (e) *For an L-space knot K we have $g_3(K) = g_4(K)$ (see [86] and [25]).*
- (f) *For any $i \in \mathbb{Z}$ we have $\text{rank} \widehat{HFK}(K, i) \leq 1$ (see [86]).*

Remark 5.23. Fiberedness of a knot admitting a lens space surgery was known to experts before the Heegaard Floer times, [86] contains an explicit proof. The proof for general L-space knots follows from the explicit description of the fact that $\text{rank} \widehat{HFK}(K, i) \leq 1$ together with the result of [20, 72] (Theorem 5.6 of the present article).

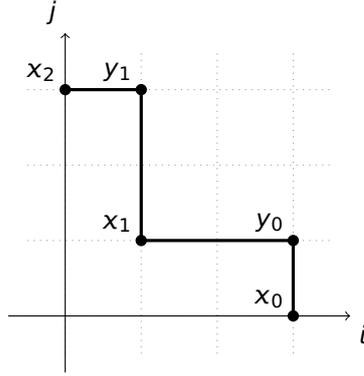


Figure 5.7: A staircase complex of a torus knot $T(3, 4)$.

Problem 53. Prove that the τ invariant (see [81]) of an L -space knot is equal to its three-genus. Notice that this proves (e). Refer to a result of Hedden [25] to prove (c).

Remark 5.24. It is easy to find a positive knot which is not an L -space knot: take the connected sum of two trefoils. There are positive knots (even fibred positive knots) which are not even concordant to a connected sum of any number of L -space knots; see [5, 15].

We will now present an algorithm for describing the CFK^∞ complex of an L -space knot based on the Alexander polynomial. The algorithm was first described by Peters [97], nowadays it is widely used.

Suppose K is an L -space knot of genus g . Let Δ be the Alexander polynomial for K , which we normalize in such a way that $\Delta(t^{-1}) = \Delta(t)$. It was showed in [86] that Δ has the following form.

$$\Delta(t) = t^{n_0} - t^{n_1} + \dots - t^{n_{2k-1}} + t^{n_{2k}},$$

where $n_0 > n_1 > \dots > n_{2k}$ and $n_0 = -n_{2k} = g$. Set

$$\begin{cases} m_0 = 0 \\ m_{2i-1} = m_{2i} & 1 \leq i \leq k \\ m_{2i+1} = m_{2i} + (n_{2i} - n_{2i+1}) & 0 \leq i \leq k-1 \end{cases}$$

Problem 54. Show that $m_{2k} = g$.

We will construct now an abstract chain complex over \mathbb{Z}_2 from the numbers n_i and m_i . The chain complex will be graded and doubly filtered. The construction is as follows.

For any $i = 0, \dots, k$ we place a generator x_i with (Maslov) grading 0 at bifiltration level (m_{2k-2i}, m_{2i}) (in the notation of Section 5.3 it is $[x_i, m_{2k-2i}, m_{2i}]$). We set $\partial x_i = 0$. For any $i = 0, \dots, k-1$ we place a generator y_i with (Maslov) grading 1 at bifiltration level $(m_{2k-2i-1}, m_{2i+1})$ (that is, $[y_i, m_{2k-2i-1}, m_{2i+1}]$). We set $\partial y_i = x_i + x_{i+1}$.

Example 5.25. For a torus knot $T(3, 4)$ we have $\Delta = t^3 - t^2 + 1 - t^{-2} + t^{-3}$ so $m_0 = 0, m_1 = 1, m_2 = 1, m_3 = 3, m_4 = 3$. The x -generators are at bifiltration levels $(0, 3), (1, 1)$ and $(3, 0)$, while the y -generators are at bifiltration level $(1, 3)$ and $(3, 1)$; see Figure 5.7.

Definition 5.26. The chain complex obtained in this way is called the *staircase complex* associated with an L -space knot K and it is denoted $St(K)$.

The staircase complex will now be tensored by $\mathbb{Z}_2[U, U^{-1}]$, where U is a formal variable. We write $St(K) \otimes_{\mathbb{Z}_2} \mathbb{Z}_2[U, U^{-1}]$ for the product. It is generated by elements $U^j x_i$ and $U^j y_i$, $j \in \mathbb{Z}$. The grading and the filtration levels are defined by requiring that multiplication by U changes the (Maslov) grading by -2 and each of the filtration levels by -1 , exactly as the action of U on the knot Floer chain complexes. The following result was described in a paper

by Peters [97] (see also [74]), but the idea that the Alexander polynomial determines the complex CFK^∞ can be traced back to [86].

Proposition 5.27. *Let K be an L -space knot. The chain complex $St(K) \otimes_{\mathbb{Z}_2} \mathbb{Z}_2[U, U^{-1}]$ is bifiltered chain homotopy equivalent to $CFK^\infty(K)$.*

Problem 55. Prove that if K and K' are L -space knots, then we have $V_m(K \# K') = \min_{k \in \mathbb{Z}} (V_k(K) + V_{m-k}(K'))$. Show that the same holds if K and K' are connected sums of L -space knots.

6. Cuspidal singularities

The scenery changes for a while. We need to recall a few facts from singularity theory.

6.1. Links of singular points

Consider a complex curve C in some connected, open set $\Omega \subset \mathbb{C}^2$. Suppose C is defined as a zero set $F^{-1}(0)$, where $F: \Omega \rightarrow \mathbb{C}$ is a holomorphic function. We will assume that F is *reduced*, which might be interpreted as requiring that the gradient of F does not vanish identically on any open subset of C .

Problem 56. The rigorous definition of ‘reduced’ reads that F is not divisible (in the ring of holomorphic functions $\mathcal{O}(\Omega)$) by any square of a non-invertible element. Prove that the two definitions are equivalent.

Definition 6.1. A point $z \in C$ is called *singular* if $\nabla F(z) = 0$.

Problem 57. Prove that if $z \in C$ is a singular point and F is reduced, then z is *isolated*, that is, there is no sequence $z_n \in C \setminus \{z\}$ of singular points converging to z .

By Tougeron’s theorem, see [117, Section 2.1], any isolated singular point is finitely presented. That is, for each singular point z there is a local analytic change of coordinates, which transforms C to a $F_{fin}^{-1}(0)$, where F_{fin} is a Taylor expansion of F at z of sufficiently high order (the original Tougeron theorem says that the order equal to the Milnor number plus one will do, but for some specific singularities a lower order expansion may be sufficient).

Let $z \in C$ be a singular point. Take a ball $B \subset \Omega$ with center z of sufficiently small radius.

Definition 6.2. The intersection $\partial B \cap C \subset \partial B$ is called the *link of singularity*.

Problem 58. Prove that the isotopy type of the link of singularity is independent of the radius of the curve, once the starting curve is sufficiently small.

Hint. The distance function to the singular point is a Morse function when restricted to C . Try showing that the restriction has no critical points on C near z , except for z itself. See also [60].

Problem 59. Prove that $C \cap B$ is homeomorphic to the cone over the link $C \cap \partial B$.

Definition 6.3. The *number of branches* of C at the singular point is the number of connected components of $B \cap C \setminus \{z\}$. A singular point is called *cuspidal* if C has precisely one branch.

Two singular points (C, z) and (C', z') are analytically equivalent if there exists a biholomorphic map of neighborhoods of z and z' in \mathbb{C}^2 , which takes locally C to C' . In general, analytic equivalence is a surprisingly complicated notion. There is a coarser equivalence, which proves very useful.

Definition 6.4. Two singular points (C, z) and (C', z') are called *topologically equivalent* if there exist small balls $B, B' \subset \mathbb{C}^2$ with centers z and z' and a homeomorphism $h: B \rightarrow B'$ that takes $C \cap B$ to $C' \cap B'$.

Problem 60. Show that two singular points are topologically equivalent if and only if their links are isotopic.

The two notions of equivalence give rise to notions of analytic and topological invariants of singular points. These are quantities associated with a singular point which are preserved under an analytic (respectively: topological) equivalence. The distinction can be quite subtle. For example, the Milnor number $\mu = \dim_{\mathbb{C}} \mathcal{O}_z / (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y})$ (here \mathcal{O}_z is the local ring and we consider its quotient over by an ideal generated by $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$) is a topological invariant. For a cuspidal singularity μ is equal to twice the genus of the link and a slightly more complicated formula calculates the Milnor number from the genera of the components of the link and the linking numbers of the components; see [61, Section 10].

On the other hand, the Tjurina number, $\tau = \dim_{\mathbb{C}} \mathcal{O}_z / (F, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y})$, whose definition looks very similar to μ , is *not* a topological invariant; see [22, Section I.1.2].

Problem 61. Show that if F is quasihomogeneous, then $\tau = \mu$.

Problem 62. Play around with some examples of F using your favorite computer algebra system (sage, macaulay, singular) and find examples of singularities which have the same topological type but different Tjurina numbers.

Hint. Take $F = x^p - y^q$ with p, q coprime and try adding to it terms of weighted degree greater than pq , where x has degree q and y has degree p .

To conclude the section we list a few different objects related to a singular point that have (almost) the same meaning.

- The Milnor number μ defined as above. By the celebrated Milnor's theorem, the map $z \mapsto F(z)/|F(z)|$ from $\partial B \setminus (C \cap \partial B)$ to S^1 is a locally trivial fibration, whose fiber has homotopy type of a wedge of μ copies of S^1 .
- The δ -invariant, whose original definition is algebraic; see [22, Section I.3.4]. For a singular point with r branches we have that $2\delta = \mu + r - 1$, a formula proved by Milnor in [61, Section 10].
- The genus of the link $g_3(C \cap \partial B)$ is equal to half the Milnor number if the link has one branch. By Kronheimer–Mrowka's result, the three-genus is also equal to the smooth four-genus of the link.

Problem 63. Establish an explicit relation between $g_3(C \cap \partial B)$ and the δ -invariant for a singular point with arbitrarily many branches. The algebraic definition of the δ -invariant is given in [22, Section I.3.4] or in [61, Section 10].

6.2. Topological classification of cuspidal singular points

For completeness we recall a topological classification of cuspidal singular points. For us it is convenient to write the classification in terms of a so-called *characteristic sequence*. A characteristic sequence is a finite sequence of numbers $(p; q_1, q_2, \dots, q_m)$ with $p > 1$, $p < q_1 < \dots < q_m$. These numbers satisfy the following relation. Set $r_0 = p$, $r_{i+1} = \gcd(r_i, q_{i+1})$. We require that the sequence r_i be strictly decreasing and $r_m = 1$. To each characteristic sequence we can associate a model singular point on a curve, which is locally parametrized as

$$\begin{aligned} x(t) &= t^p \\ y(t) &= t^{q_1} + t^{q_2} + \dots + t^{q_m}. \end{aligned}$$

Theorem 6.5. *The characteristic sequence is a complete invariant of the topological type of cuspidal singular points. That is, any cuspidal singular point is topologically equivalent to precisely one model singularity.*

The number m is called the *length* of the characteristic sequence. There are several alternative ways of encoding a characteristic sequence. For example, there are so-called Newton pairs, and Puiseux or characteristic pairs (both Newton pairs and Puiseux pairs might have slightly different meaning), which are sequences of pairs of integers. The quantity m is also the length of such a sequence and so we will often refer to m as the number of Puiseux pairs. Note however, that the multiplicity sequence, see [9], might be much longer than the characteristic sequence.

The isotopy class of the link of singularity is also an invariant of topological type, and in fact, it is also a complete invariant. There is an explicit algorithm for determining the link from the characteristic sequence; see [14]. We record one basic example for future reference.

Example 6.6. If $m = 1$, the characteristic sequence is $(p; q)$ for some coprime integers with $0 < p < q$. The link of singularity is the torus knot $T(p, q)$.

6.3. Semigroup of a singular point

Let (C, z) be a singular point of a plane curve. For any complex polynomial G , which does not vanish on any of the components of C containing z (component in the analytic sense), we can define the local intersection index $C \cdot_z G^{-1}(0)$.

Example 6.7. Suppose z is cuspidal. By the Puiseux theorem there exists a local parametrization $t \mapsto (x(t), y(t))$ of C near z , such that $z = (x(0), y(0))$. Then the local intersection index is the order at $t = 0$ of the map $t \mapsto G(x(t), y(t))$.

Problem 64. Suppose $z = (0, 0)$ and $C = \{F \equiv 0\}$ with $F = x^p - y^q$. Show that a number $l \geq 0$ can be obtained as $C \cdot_z G^{-1}(0)$ if and only if l can be presented as $ip + jq$, where $i, j \geq 0$ are integers. For $l = ip + jq$ write explicitly a polynomial G such that $C \cdot_z G^{-1}(0) = l$.

We have the following notion.

Definition 6.8. The *semigroup* of a singular point $S(z)$ is a semigroup of $\mathbb{Z}_{\geq 0}$ whose elements are local intersection indices $C \cdot_z G^{-1}(0)$ as G ranges through all the polynomials $\mathbb{C}[x, y]$ that do not vanish on any of the components of C containing z . By convention, zero is always considered as an element of $S(z)$: it corresponds to a polynomial G that does not vanish at z .

Problem 65. Show that S is in fact a semigroup.

Problem 66. Show that the smallest non-zero element of the semigroup is the multiplicity of a singular point.

The notion of the semigroup as defined here is useful mostly for cuspidal singular points. If z has $r > 1$ branches, it might be more natural to consider a semigroup of \mathbb{Z}^r , whose elements are vectors formed by local intersection indices with the branches. There is a significant difference between the cuspidal and non-cuspidal case. In the present notes we focus mostly on the cuspidal case.

Theorem 6.9 (see e.g. [112, Chapter 4]). *The semigroup of a cuspidal singular point z has the following properties.*

- The gap set $G := \mathbb{Z}_{\geq 0} \setminus S$ has cardinality $\mu/2$. Here μ is the Milnor number.
- The maximal element of G is equal to $\mu - 1$.
- The semigroup has the following symmetry property: for any $x \in \mathbb{Z}$, either $x \in S$, or $2g - 1 - x \in S$, but never both.

Problem 67. Deduce the first two properties in the statement of Theorem 6.9 from the third one.

Problem 68. Prove elementarily that if S is a semigroup generated by p and q , then the maximal element that does not belong to the semigroup is $(p-1)(q-1)-1$.

Problem 69. Suppose S is a semigroup of $\mathbb{Z}_{\geq 0}$ such that $G = \mathbb{Z}_{\geq 0} \setminus S$ is finite. Assume that S has three generators (p, q, r) . Try finding an explicit formula for the maximal element of G if $p = 2$ or $p = 3$ and see how hard it is. This shows that the second property of Theorem 6.9 is very special. See [99] for a detailed discussion on numerical semigroups.

We have the following fact first established in [11]. We refer also to [112, Chapters 4,5].

Theorem 6.10. Let z be a cuspidal singular point with a semigroup S . Let $G = \mathbb{Z}_{\geq 0} \setminus S$ be the gap set. Then

$$1 + (t-1) \sum_{r \in G} t^r$$

is the Alexander polynomial of the link of the singular point z .

The result is unexpected and shows very deep relations between singularity theory and knot theory, see [11] for more details. Nevertheless, the theorem is not hard to prove, since we exactly know which links can arise from cuspidal singularities. They are, see [10, 113], iterated cables on torus knots. Both the link of the singularity and the semigroup can be determined from the Puiseux pairs of singular points. The proof of Theorem 6.10 consists of calculating both sides in terms of Puiseux pairs and in fact, the only non-trivial result that is used is the formula for the Alexander polynomial of a cable. On the other hand we have just shown that the semigroup is a topological invariant of a singular point.

6.4. Links of singularities as L-space knots

In [26] Hedden proved the following result.

Theorem 6.11. The link of a cuspidal singularity is an L-space knot.

Suppose z is a cuspidal singular point with semigroup S and link K . The semigroup determines the Alexander polynomial by Theorem 6.10. As K is an L-space knot, the Alexander polynomial of K determines the chain complex CFK^∞ . This chain complex determines the concordance invariants V_m . Therefore, the numbers V_m can be calculated directly from the semigroup S . An explicit computation is not hard.

Theorem 6.12 (compare [7, Proposition 4.6]). We have $V_{g+m} = \#\{j \geq m : j \notin S\}$, where g is the genus of the knot K .

Problem 70. Use Theorem 6.10 and the explicit algorithm for calculating CFK^∞ of an L-space knot (see Proposition 5.27 and the algorithm above it) to prove Theorem 6.12.

In conjunction with Large Surgery Theorem 5.18 this result will allow us to calculate d -invariants of large surgeries on links of cuspidal singularities from the semigroup only.

Remark 6.13. Even if Theorem 6.12 is easy to believe and rather straightforward to prove, it sets a right perspective. The semigroup is a natural object to study when one is interested in applications of Heegaard Floer techniques in singularity theory.

7. Rational cuspidal curves and beyond

7.1. What is a rational cuspidal curve?

We now pass to considering complex curves in $\mathbb{C}P^2$. Let $C \subset \mathbb{C}P^2$ be an *irreducible* curve, that is, a curve which cannot be presented as a union of two curves $C_1 \cup C_2$. Put differently, an irreducible curve is a curve that can be realized as a zero set of a homogeneous polynomial

F which is irreducible in $\mathbb{C}[x, y, z]$. The *degree* of the curve C is the degree of a reduced homogeneous polynomial whose zero set is C .

If C is a smooth curve of degree d , its genus is determined by d , namely $g(C) = \frac{(d-1)(d-2)}{2}$. For singular curves the notion of genus can be generalized in many non-equivalent ways. The most useful to us is the notion of a geometric genus. To introduce it, recall that any complex curve in $\mathbb{C}P^2$ admits a so called *normalization*. This is a smooth complex curve Σ together with a complex map $\pi: \Sigma \rightarrow C$, such that the inverse image of each of the singular points is finite and the preimage of each smooth point consists of a single point. It is not hard to show that a normalization exists and is well defined up to a biholomorphism.

Definition 7.1. The *geometric genus* $p_g(C)$ is the genus of the normalization Σ . A curve C is called *rational* if its geometric genus is zero. A curve is called *rational cuspidal* if it is rational and all its singular points (if any) are cuspidal.

Problem 71. Prove that C is rational cuspidal if and only if it is homeomorphic to the sphere S^2 .

For completeness, we recall a classical numerical formula for the geometric genus.

Theorem 7.2. Suppose C has degree d and singular points z_1, \dots, z_n . Let $\delta_1, \dots, \delta_n$ be the δ -invariants of z_1, \dots, z_n (for a cuspidal singularity the δ -invariant is equal to the genus of the link; if z_i has $r_i > 1$ branches, then $2\delta_i = \mu_i + r_i - 1$). Then

$$p_g(C) = \frac{1}{2}(d-1)(d-2) - \sum_{i=1}^n \delta_i.$$

Milnor in [61, Section 10] attributes Theorem 7.2 to Serre, however at least some variant of it was known already in the XIXth century.

7.2. A quick tour of rational cuspidal curves

Rational cuspidal curves have been an object of interest at least since the end of the XIXth century. Before we state one of the most important conjectures on rational cuspidal curves, we give a definition; see [24, Section 1.4].

Definition 7.3.

- A *rational map* between two algebraic irreducible varieties $f: X \rightarrow Y$ is an equivalence class of pairs (U, f_U) , where U is a Zariski open subset of X and $f_U: U \rightarrow Y$. Two pairs (U, f_U) and $(U', f_{U'})$ are said to be equivalent if they agree on $U \cap U'$.
- A *birational map* $f: X \rightarrow Y$ is a rational map that admits a rational inverse, that is, a rational map $g: Y \rightarrow X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$, where the equalities are understood as equivalences of rational maps.

A reader not familiar with algebraic geometry might be worried that a rational map is defined only on an open subset of X . The key word here is ‘Zariski open’. The Zariski topology is completely different from the metric topology. Open sets are basically complements of hypersurfaces, so an open set in Zariski topology means an open-dense subset of X in the metric topology, whose complement is of complex codimension at least 1.

Example 7.4. A blow-up and blow-down are birational maps.

Example 7.5. It was proved already by Zariski, see [114], that any birational map between two algebraic surfaces is a sequence of blow-ups and blow-downs.

Now we pass to an important definition.

Definition 7.6. A curve $C \subset \mathbb{C}P^2$ is called *rectifiable* if there exists a birational map $f: \mathbb{C}P^2 \rightarrow \mathbb{C}P^2$ such that the (closure of) the image $f(C)$ is a straight line.

Problem 72. Show that a curve C given by $x^3 = y^2z$ in homogeneous coordinates $[x : y : z]$ in $\mathbb{C}P^2$ is rectifiable.

In 1928 Coolidge [13] stated a conjecture, which was given its final shape by Nagata [66].

Conjecture 7.7 (The Coolidge–Nagata conjecture). Any rational cuspidal curve is rectifiable.

The conjecture eluded all approach until 2015, when two mathematicians, Koras and Palka, found a brilliant proof relying on the minimal model program.

Theorem 7.8 (see [39, 94]). *The Coolidge–Nagata conjecture is true. That is, every rational cuspidal curve in $\mathbb{C}P^2$ can be transformed into a line by means of birational transformations of $\mathbb{C}P^2$.*

The meaning of the conjecture is that every rational cuspidal curve can be constructed by taking a line and applying a sequence of blow-ups and blow-downs. This does not solve the problem of classifying all the rational cuspidal curves, because the configurations of blow-ups and blow-downs might be rather complicated.

Problem 73 (Open). Use methods of Koras and Palka to prove that every rational cuspidal curve in a Hirzebruch surface is rectifiable. See [63, 64, 8] for more on rational cuspidal curves in Hirzebruch surfaces.

Another problem concerning rational cuspidal curves is to establish bounds for the number of possible singular points. The following conjecture is due to Orevkov. It circulated among the experts for a long time and was stated explicitly in a paper by Piontkowski [98].

Conjecture 7.9. Any rational cuspidal curve $C \subset \mathbb{C}P^2$ has at most four singular points. Moreover, there is only one curve (of degree 5) that has precisely four singular points.

For a long time the best upper bound was 8 [107]. Recently Palka improved this bound to 6, see [93].

There is another conjecture due to Flenner and Zajdenberg [18], called the Rigidity Conjecture. Introducing all the terminology needed to state it is beyond the scope of the present article, so we will be rather informal. Suppose $C \subset \mathbb{C}P^2$ is a rational cuspidal curve. We resolve the singularities of C to obtain a surface V together with a rational map $\pi: V \rightarrow \mathbb{C}P^2$. The inverse image $D = \pi^{-1}(C)$ (for algebraic geometers: we take a reduced scheme structure on D) is a simple normal crossing divisor, that is, it is a union of holomorphic spheres intersecting transversally such that no self-intersections are allowed and triple intersection points are excluded. Such a resolution (V, D) always exists, see [9, 22, 14] or almost any book on complex plane curves.

One studies the infinitesimal deformations of the pair (V, D) in the spirit of Kodaira and Spencer [38]. There is a sheaf, $\Theta_V\langle D \rangle$ of complex vector fields on V that are tangent to D . It turns out, see [18], that this sheaf controls the deformations of the pair (V, D) , that is, h^1 of this sheaf is the space of infinitesimal deformations of the pair (V, D) and h^2 is the space of obstructions to the deformations. If $h^2(\Theta\langle D \rangle) = 0$, the deformations are unobstructed, because higher obstructions (h^i for $i > 2$) vanish for dimensional reasons. Now the Flenner–Zajdenberg rigidity conjecture states that $h^2(\Theta\langle D \rangle) = 0$, that is, infinitesimal deformations are unobstructed. In most interesting cases $h^0(\Theta\langle D \rangle) = 0$, so $\chi(\Theta\langle D \rangle) \leq 0$ (recall that $\chi = h^0 - h^1 + h^2$). On the other hand, the Riemann–Roch theorem for surfaces tells us that $\chi(\Theta\langle D \rangle) = K(K + D)$, so the Rigidity Conjecture implies that $K(K + D) \leq 0$, but the converse implication does not necessarily hold, which is one of the reasons why the conjecture is so difficult. It is well-known to the experts that the Rigidity Conjecture implies the Coolidge–Nagata conjecture, but again, the converse implication is not true; see also [93] for a more detailed discussion.

7.3. Partial results on classification

Rational cuspidal curves with logarithmic Kodaira dimension less than 2 have already been classified, see the introduction in [17] for a concise summary of the results. The logarithmic

Kodaira dimension, $\bar{\kappa}$, defined in [32], is an invariant of a complement $V \setminus D$, where V is a projective surface and D a divisor on it. If V is a surface, then $\bar{\kappa}(V \setminus D) \in \{-\infty, 0, 1, 2\}$. It is a result of Wakabayashi [111], that if $C \subset \mathbb{C}P^2$ is a rational cuspidal curve such that $\bar{\kappa}(\mathbb{C}P^2 \setminus C) \leq 0$, then C has at most one singular point, moreover if $\bar{\kappa}(\mathbb{C}P^2 \setminus C) = 1$, then C has at most two singular points.

The classification of rational cuspidal curves such that $\bar{\kappa}(\mathbb{C}P^2 \setminus C) = -\infty$ was achieved by Kashiwara [36]. The case $\bar{\kappa}(\mathbb{C}P^2 \setminus C) = 0$ was excluded by Tsunoda in [108], another reference is [76]. Classification of curves with $\bar{\kappa}(\mathbb{C}P^2 \setminus C) = 1$ was started by Kishimoto [37] and completed by Tono in [106].

The case $\bar{\kappa}(\mathbb{C}P^2 \setminus C) = 2$ is the hardest. There is a program of Palka and Peřka on classifying all rational cuspidal curves that satisfy the Flenner–Zaidenberg Rigidity conjecture; see [95] for the first important results in that direction.

On the other side, somehow setting aside the logarithmic Kodaira dimension, in [16] an attempt was made to classify rational cuspidal curves. The result was only the first step, namely the following result is proved in [16].

Theorem 7.10. *Suppose C is a rational cuspidal curve in $\mathbb{C}P^2$ having precisely one singular point. Assume additionally that this singular point has a single Puiseux pair (p, q) . Then the pair (p, q) belongs to one of the following list. Moreover, each pair below can be realized by a rational cuspidal curve.*

- (a) $(d-1, d)$ for any $d > 1$,
- (b) $(d/2, 2d-1)$ for any even $d > 1$,
- (c) (ϕ_{j-2}^2, ϕ_j^2) for j odd and $j \geq 5$, where ϕ_i are Fibonacci numbers normalized in such a way that $\phi_0 = 0, \phi_1 = 1$,
- (d) (ϕ_{j-2}, ϕ_{j+2}) for $j \geq 5$ odd,
- (e) $(\phi_4, \phi_8 + 1) = (3, 22)$,
- (f) $(2\phi_4, 2\phi_8 + 1) = (6, 43)$.

Problem 74. Determine the degree of C in each of the cases (c)–(f). Cases (a) and (b) are trivial.

In [2], based on the thesis of Tiankai Liu [47], Bodnár gave an analogue of Theorem 7.10 for rational cuspidal curves with one singular point such that the singular point has two Puiseux pairs. The result is more complicated.

7.4. The tubular neighborhood of a rational cuspidal curve

We pass to applications of Heegaard Floer theory to rational cuspidal curves.

Let $C \subset \mathbb{C}P^2$ be a rational cuspidal curve of degree d . We aim to construct a ‘tubular’ neighborhood of C in $\mathbb{C}P^2$. The word ‘tubular’ is in quotation marks, because C is not locally flat and cannot have a product neighborhood. However, the following, rather obvious, construction will fit well into our applications.

For any singular point z_i of C take a small ball B_i with center z_i . The complement $C \setminus \bigcup B_i$ is a smooth curve so we take product neighborhood N_0 . We will require that N_0 is thin as compared to the radii of all the B_i . Set $N = N_0 \cup \bigcup B_i$. Clearly N is an open set containing C . Alternatively we could define N as a set of points at distance less than ε of C for $\varepsilon > 0$ sufficiently small; this leads to essentially the same space N . However, the first construction has an advantage, namely the following Lemma is easy to notice.

Lemma 7.11. *Let $Y = \partial N$. Let z_1, \dots, z_n be the singular points of C and K_1, \dots, K_n its links. Set $K = K_1 \# \dots \# K_n$. Then $Y = S_{d^2}^3(K)$.*

Problem 75. Prove Lemma 7.11 for $n = 1$ (hint: notice that d^2 is the self-intersection of C).

For $n > 1$ the proof of Lemma 7.11 is given in [7, Section 3].

Let us consider $W = \mathbb{C}P^2 \setminus N$. The homology of W can be easily calculated: notice that C is a generator of $H_2(\mathbb{C}P^2; \mathbb{Q})$, removing C from $\mathbb{C}P^2$ should yield a rational homology ball. This indeed is so.

Problem 76. Prove that $H_k(W; \mathbb{Q}) = 0$ if $k > 0$.

Problem 77. Calculate the \mathbb{Z} -homologies of W .

We pass to describing Spin^c structures on W , with the aim to calculate which Spin^c structures on $Y = \partial N$ extend over W . The three-manifold Y is a d^2 surgery on K . Therefore, we can enumerate Spin^c structures on Y by integers $m \in [-d^2/2, d^2/2]$ as in Proposition 5.17 above.

Problem 78. Show that the Spin^c structure s_m on Y extends to a Spin^c structure t_m on N such that $\langle c_1(t_m), C \rangle + d^2 = 2m$.

Suppose now a Spin^c structure s_m on Y extends to a Spin^c structure t'_m on W . The Spin^c structures t_m and t'_m on N and W glue together to a Spin^c structure t''_m on $\mathbb{C}P^2$. Now $\mathbb{C}P^2$ is a closed simply connected four-manifold. By Corollary 2.9 it follows that $c_1(t''_m) = (2j + 1)[H]$ for some $j \in \mathbb{Z}$, where $[H]$ is the generator of $H^2(\mathbb{C}P^2; \mathbb{Z})$. In particular

$$\langle c_1(t_m), C \rangle = \langle c_1(t''_m), C \rangle = (2j + 1)d.$$

Applying Proposition 5.17 we obtain the following statement.

Lemma 7.12. *If a Spin^c structure s_m on Y extends over W , then $m = \frac{1}{2}(d^2 - (2j + 1)d)$ for some $j \in \mathbb{Z}$.*

7.5. Heegaard Floer homology applied to rational cuspidal curves

Let us now gather all pieces of a puzzle to restrict the Alexander polynomials of links of singular points a rational cuspidal curve. We suppose first that C is a rational cuspidal curve of degree d with one singular point z , whose link is K and whose semigroup is S .

- The boundary of the tubular neighborhood of C is $S_{d^2}^3(K)$.
- K is an algebraic knot, hence an L-space knot.
- The V_m invariants of K can be calculated from the semigroup S .
- The genus of K is $\frac{1}{2}(d - 1)(d - 2)$. The surgery coefficient d^2 is greater than twice the genus.
- The Large Surgery Theorem applies. We can express the d -invariants of Y in terms of the semigroup.
- On the other hand Y bounds a rational homology ball W . Hence $d(Y, s_m) = 0$ for every Spin^c structure s_m on Y that extends over W .
- The Spin^c structures on Y that extend over W were calculated in Lemma 7.12 above. We get restrictions for the distribution of elements in the semigroup S .

These restrictions can be stated as follows.

Theorem 7.13 (see [7]). *For any $j = 0, \dots, d - 2$ we have $\#S \cap [0, jd + 1] = \frac{1}{2}(j + 1)(j + 2)$.*

Problem 79. Using the itemized list, prove Theorem 7.13.

The case $n > 1$ is similar; the new technical difficulties are rather minor. Suppose C is a rational cuspidal curve of degree d with singular points z_1, \dots, z_n , whose links are K_1, \dots, K_n respectively and the associated semigroups are S_1, \dots, S_n . Set $K = K_1 \# \dots \# K_n$. Then $Y = \partial N$ is $S_{d^2}^3(K)$ as states Lemma 7.11 above. However, as mentioned in Lemma 5.22(a) K has no chances to be an L-space knot if $n > 1$, in fact, K is not prime. Luckily K is a connected sum of L-space knots K_1, \dots, K_n , hence by the Künneth formula (Proposition 5.10) we have $CKF^\infty(K) = CKF^\infty(K_1) \otimes \dots \otimes CKF^\infty(K_n)$. The Künneth formula allows us to express the V_m invariants of K in terms of the V_m invariants of the summands. Acting as in Problem 55 we obtain

$$V_m(K) = \min_{m_1 + \dots + m_n = m} V_{m_1}(K_1) + \dots + V_{m_n}(K_n).$$

Now each of the $V_{m_i}(K_i)$ can be expressed from the semigroup of the singular point z_i . Putting things together and acting as in the case $n = 1$ we arrive at the following result.

Theorem 7.14 (see [7]). *For any $j = 0, \dots, d - 2$ we have*

$$\min_{k_1 + \dots + k_n = j + 1} \sum_{i=1}^n \#S_i \cap [0, k_i] = \frac{1}{2}(j + 1)(j + 2),$$

7.6. Strength and weakness of Theorems 7.13 and 7.14

Theorem 7.13 has proved very useful in classifying rational cuspidal curves with one singular point. It is possible to give a full list of possible rational cuspidal curves with one singular point having one Puiseux pair (this is equivalent to saying that the link is a torus knot), using essentially Theorem 7.13. This classification was first done in [16], the proof using Theorem 7.13 is considerably simpler.

Problem 80. Show that there is a value $d_0 > 0$ such that for any $d > d_0$ there are no rational cuspidal curves of degree d with one Puiseux pair (p, q) such that $p < q$ and $p \in (d/2, d - 1)$.

Problem 81. Use Theorem 7.13 to prove that if a rational cuspidal curve of degree d has one singular point, then its multiplicity is at least $d/3$. The original proof of Matsuoka and Sakai [59] uses the BMY inequality.

As it was shown in [7, Section 6], for $n = 1$ and a general number of Puiseux pairs, the restriction of Theorem 7.13 has approximately the same strength as the spectrum semicontinuity property (see [16] for more details). There are relatively few cases when Theorem 7.13 gives an obstruction, while the spectrum semicontinuity does not. There are also very few cases when the opposite holds.

Surprisingly, for $n \geq 2$ the situation changes and Theorem 7.14 is not that strong anymore. A potential problem was discovered by Bodnár and Némethi [3] (see also [15]). Before we state it, we give an example.

When trying to classify all rational cuspidal curves of degree 5 with two singular points, both having multiplicity 2, one finds that the genus formula (Theorem 7.2) implies that we might have three cases: either the singular points are $(2; 3), (2; 11)$, or $(2; 5), (2; 9)$, or $(2; 7), (2; 7)$. Such classification was already known long before; see [62].

Problem 82. Prove that in each of the three cases, if S_1 and S_2 denote the corresponding semigroups, we have

$$\min_{i+j=k} \#S_1 \cap [0, i] + \#S_2 \cap [0, j] = \begin{cases} \lfloor (k + 1)/2 \rfloor & k \leq 12 \\ k - 6 & k \geq 12. \end{cases}$$

Therefore Theorem 7.14 is unable to distinguish between the three cases. As the curve of degree 5 with singular points $(2; 5)$ and $(2; 9)$ actually exists, we cannot obstruct any of the remaining two cases. On the other hand, these remaining two cases do not exist.

The deeper reason was discovered in [3, Section 5]. To describe it we introduce a bit of notation. Namely, to any cuspidal singular point z we associate its *multiplicity sequence* M_z . For a set of singular points z_1, \dots, z_n the union $\mathcal{M} = M_1 \cup \dots \cup M_n$ is an unordered tuple of integers greater than 1 (each integer can enter several times in the union). We say that $\mathcal{M} = \mathcal{M}'$ if for each integer $i \geq 2$ the number of times i appears in \mathcal{M} is equal to the number of times it appears in \mathcal{M}' . We have the following result.

Theorem 7.15 (see [3, Theorem 5.1.3]). *Suppose z_1, \dots, z_n and $z'_1, \dots, z'_{n'}$ are two collections of singular points. Let S_1, \dots, S_n and $S'_1, \dots, S'_{n'}$ be corresponding semigroups and M_1, \dots, M'_n be the multiplicity sequences. Set $\mathcal{M} = M_1 \cup \dots \cup M_n$ and $\mathcal{M}' = M'_1 \cup \dots \cup M'_{n'}$. If $\mathcal{M} = \mathcal{M}'$, then for every $k \in \mathbb{Z}$ we have*

$$\min_{i_1 + \dots + i_n = k} \sum_{j=1}^n \#S_j \cap [0, i_j) = \min_{i'_1 + \dots + i'_{n'} = k} \sum_{j=1}^{n'} \#S'_j \cap [0, i'_j).$$

The result greatly limits the applicability of Theorem 7.14 when $n > 1$.

Remark 7.16. There exists a Heegaard Floer proof of the fact that a rational cuspidal curve of degree 5 cannot have two singular points $(2; 3)$ and $(2; 11)$, neither can it have two singular points $(2; 7)$ and $(2; 7)$; see [62, Section 6.1.3]. The proof involves involutive Floer theory as developed by Hendricks and Manolescu [27], which is beyond the scope of the present article. See [6] for details.

7.7. Relation to the FLMN conjecture

In 2006, Fernández de Bobadilla, Luengo Velasco, Melle Hernández and Némethi suggested the following conjecture.

Conjecture 7.17 (see [17]). Let $C \subset \mathbb{C}P^2$ be a rational cuspidal curve of degree d . Let K be the connected sum of links of singularities of C . Write the Alexander polynomial of K as $\Delta_K(t) = 1 + (t-1)\delta + (t-1)^2 Q(t)$ for some polynomial $Q(t)$ and let c_j be the coefficient of Q at $t^{d(d-3-j)}$. Then for $j = 0, \dots, d-3$

$$c_j \leq \frac{1}{2}(j+1)(j+2).$$

Moreover, if C has precisely one singular point, then $c_j = \frac{1}{2}(j+1)(j+2)$ for all $j = 0, \dots, d-3$.

Problem 83. Show that δ in the statement of Conjecture 7.17 is always equal to $\frac{1}{2}(d-1)(d-2)$.

Before we discuss the relation of Conjecture 7.17 to Theorem 7.14 in greater detail, let us first say something about the motivation of the conjecture. Namely, in a series of papers, Némethi and Nicolaescu studied the relation of the Seiberg–Witten invariants of normal surface singularities and their geometric genus p_g . In [69] they stated a conjecture, called the Seiberg–Witten invariant conjecture. The conjecture was verified for many families of surface singularities in [69, 70, 71]. However, in [50] it was shown that superisolated surface singularities are expected to satisfy the opposite inequality to the one conjectured by Némethi and Nicolaescu. Superisolated surface singularities were introduced by Luengo in [49] and are tightly related to rational cuspidal curves. In fact, each rational cuspidal curve C gives rise to a superisolated surface singularity whose link is $S^3_{-d}(K)$, where d is the degree of the curve C and K is the connected sum of links of singular points of C . Conjecture 7.17 arose as a translation the Seiberg–Witten invariant conjecture for superisolated surface singularities into the language of rational cuspidal curves.

Remark 7.18. It is no surprise that the Alexander polynomial of K appears in the context of a conjecture related to Seiberg–Witten invariants of the link $S^3_{-d}(K)$. In fact, the relation of Seiberg–Witten invariants with the Reidemeister–Turaev torsion (see [110] and references

therein) allows to calculate the Seiberg–Witten invariants of $S^3_{-d}(K)$ from the Alexander polynomial of K ; see e.g. [17, Formula (3)].

Now we pass to the relations of Conjecture 7.17 to Theorem 7.14. We begin with the easy case.

Problem 84. Prove that if C has precisely one singular point, then Conjecture 7.17 is equivalent to Theorem 7.14.

The case that C has two singular points is more complicated.

Theorem 7.19 (see [3, 68]). *If C has two singular points, then Conjecture 7.17 follows from Theorem 7.14.*

However, if C has three or more singular points, Conjecture 7.17 is false. The following example is elaborated in [3].

Problem 85. Let C be a rational cuspidal curve of degree 8 with singular points $(6; 7)$, $(2; 9)$ and $(2; 5)$. Prove that C violates Conjecture 7.17.

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