Winter Braids Lecture Notes

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Ordered groups, knots, braids and hyperbolic 3-manifolds

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1. Introduction

These notes are based on a minicourse of three one-hour lectures given at the Winterbraids conference at the University of Caen from February 27 to March 2, 2017. Here are the titles of my lectures:

Lecture 1: Introduction to ordered groups

Lecture 2: Ordering knot groups; fibred knots and surgery

Lecture 3: Braids, $Aut(F_n)$ and minimal volume hyperbolic 3-manifolds.

It's more convenient to group the material by subject, so there are more than three sections to these notes. I posed certain problems for the audience to solve on their own, and leave these as exercises for the reader. I have added some material which was not in my talks, but in the spirit of lecture notes, some things will be discussed rather informally and certainly not everything of interest will be covered. Much of the discussion involves joint work at various times with Steve Boyer, Adam Clay, Eiko Kin, Thomas Koberda, Bernard Perron, Bert Wiest, Jun Zhu and others.

I also want to thank Patrick Dehornoy – whose birthday we are celebrating at this conference – for his amazing work on ordering braid groups (and, of course many other things!). That's what got me started on, and somewhat addicted to, orderable groups and their connections with topology. Also a great friendship.

2. Ordered groups

A left-ordered group $(G, <)$ is a group G and a strict total ordering $<$ of its elements such that $g < h$ implies $fg < fh$ for all $f, g, h \in G$. If such an ordering exists, a group G is said to be *left-orderable*.

Left-orderable groups are also right-orderable, but by a possibly different ordering. If a group has a strict total ordering $<$ which is both right- and left-invariant, we call it *bi-ordered* (in the classic literature, it is simply called "ordered"). There are many examples of orderable groups which arise in nature:

- $\mathbb{Z}^n$ (as an additive group) is bi-orderable.

For $n = 1$ there are exactly two bi-orderings, for $n \geq 2$ there are uncountably many.

- Free groups are bi-orderable.

We will discuss explicit orderings of free groups later.

- Braid groups are left-orderable [8]. But B_n is not bi-orderable if $n \geq 3$.

- Pure braid groups are bi-orderable [24].

Our proof used the Artin “combing” technique expressing the pure braid group as a semidirect product of free groups. The result also follows from a more general result that residually torsion-free nilpotent groups are bi-orderable.

- Surface groups (that is, fundamental groups of surfaces) are bi-orderable, except the Klein bottle group $\langle a, b | \, a^2 = b^2 \rangle$ which is only left-orderable, and the projective plane’s group $\mathbb{Z}/2\mathbb{Z}$ which is not even left-orderable.

Bi-orderability of surface groups was long known for the orientable case and shown in [23] for the non-oriented (hyperbolic) cases, despite claims in the literature [18] that none of the nonorientable surface groups are bi-orderable.

Consider the set $Homeo^+(\mathbb{R})$ of order-preserving homeomorphisms of the reals. It is a group under composition of homeomorphisms.

- $Homeo^+(\mathbb{R})$ is left-orderable.

For example, take your favourite enumeration $\mathbb{Q} = \{x_1, x_2, \ldots\}$ of the rationals and compare functions $f, g : \mathbb{R} \to \mathbb{R}$ by declaring $f \prec g$ iff $f(x_i) < g(x_i)$ at the first i at which $f(x_i)$ and $g(x_i)$ differ (this will exist if $f \neq g$).

Moreover, if G is a *countable* left-orderable group, then G is isomorphic with a subgroup of $Homeo^+(\mathbb{R})$. So $Homeo^+(\mathbb{R})$ is universal in the sense that it “contains” all countable left-orderable groups. This provides an intimate connection between ordered groups and dynamics. See for example [9].

Clearly left- or bi-orderability is inherited by subgroups. By using a lexicographic ordering, one can also see easily that

- A direct product of bi-orderable groups is bi-orderable; similar for left-orderable.

The following theorem [27] is more difficult.

- A free product of bi-orderable groups is bi-orderable; similar for left-orderable.

The Lie group $SL_2(\mathbb{R})$ of 2×2 matrices with real entries and determinant 1 has a universal covering group $\widetilde{SL}_2(\mathbb{R})$. It is one of the eight Thurston geometries for 3-manifolds.

- $\widetilde{SL}_2(\mathbb{R})$ is left-orderable.

This follows because $\widetilde{SL}_2(\mathbb{R})$ embeds in $Homeo^+(\mathbb{R})$. Indeed $SL_2(\mathbb{R})$ acts on the circle by orientation-preserving homeomorphisms. For example it acts on lines through the origin in $\mathbb{R}^2$. Another action is by fractional linear transformations of $\mathbb{R} \cup \infty$. Topologically, $SL_2(\mathbb{R})$ is homeomorphic with $S^1 \times D^2$, where D^2 is the open unit disk. Its universal cover $\widetilde{SL}_2(\mathbb{R})$ is an infinite cyclic cover and is a group which acts on $\mathbb{R}$ by orientation-preserving homeomorphisms.

2.1. Special algebraic properties of ordered groups

Proposition 2.1.1. *Left-ordered groups G are torsion-free, that is there are no elements of finite order.*

Indeed, suppose $g \in G$ and $g \neq 1$. If $1 < g$, then $1 < g < g^2 < g^3 < \cdots$. All powers of g are greater than 1. Similarly, if $g < 1$, no power of g can be the identity.

One easily checks that in a bi-ordered group, one can multiply inequalities: $g < h$ and $g' < h'$ imply $gg' < hh'$. (This is not true in general for left-ordered groups.) In particular, if in a bi-ordered group we have $g < h$, we conclude $g^2 < h^2$, then $g^3 < h^3$, etc. That is if g and h are unequal, then their powers g^n and h^n are also unequal.

Proposition 2.1.2.

In a bi-ordered group: $g^n = h^n$ for some $n > 0 \implies g = h$.

Proposition 2.1.3.

In a bi-ordered group, if g commutes with $h^n, n \neq 0$, then g commutes with h.

Exercise 2.1.4. *Prove this (hint: compare g with $h^{-1}gh$). Argue more generally that if any two nonzero powers of g and h commute, then g and h also commute, in a bi-orderable group.*

2.2. Group rings

Recall that the group ring RG of a group G, with coefficients in a ring R, consists of formal finite linear combinations of group elements with R coefficients. A typical element is of the form

$$\sum_{i=1}^{m} r_i g_i \quad r_i \in R, g_i \in G.$$

Multiplication is defined as for polynomials:

$$(\sum_{i=1}^{m} r_i g_i)(\sum_{j=1}^{n} s_j h_j) = \sum_{i,j} r_i s_j g_i h_j$$

Suppose a group G has a torsion element, say for example $g \in G$ has order 5, then we have an equation:

$$(1 + g + g^2 + g^3 + g^4)(1 - g) = 1 - g^5 = 0.$$

The two terms on the left are nonzero in $\mathbb{Z}G$, yet their product equals zero. Such elements are called *zero divisors*.

Our example illustrates that if G contains elements of finite order, then $\mathbb{Z}G$ has zero divisors. The converse (amazingly!) is still a matter of conjecture, attributed to Kaplansky in the late 1940's.

Zero divisor Conjecture: If the ring R has no zero divisors and G is a torsion-free group, then RG has no zero divisors.

This is unsolved, even for the case $R = \mathbb{Z}$.

Theorem 2.2.1.

Left-orderable groups satisfy the zero-divisor conjecture, that is, if R has no zero divisors and G is left-orderable, then RG has no zero divisors.

Proof: Consider a product $(\sum_{i=1}^{m} r_i g_i)(\sum_{j=1}^{n} s_j h_j) = \sum_{i,j} r_i s_j g_i h_j$, where we assume that the r_i and s_j are all nonzero, the g_i are distinct and the h_j are written in strictly ascending order, with respect to a given left-ordering of G.

At least one of the group elements $g_i h_j$ on the right-hand side is minimal in the left-ordering. If $j > 1$ we have, by left-invariance, that $g_i h_1 < g_i h_j$ and $g_i h_j$ is not minimal. Therefore we must have $j = 1$.

On the other hand, since we are in a group and the g_i are distinct, we have that $g_i h_1 \neq g_k h_1$ for any $k \neq i$. We have established that there is exactly one minimal term on the r.h.s. It follows that it survives any cancellation, and so the r.h.s. cannot be zero (because $r_i s_1 \neq 0$). Thus RG has no zero divisors.

Exercise 2.2.2. *Assume R has no zero divisors and G is left-orderable. Show that the only units (invertible elements) of RG are "monomials" of the form rg, where $g \in G$ and r is an invertible element of R. These are the so-called trivial units.*

An idempotent *in a ring is an element x such that $x^2 = x$ but $0 \neq x \neq 1$. Show that idempotents are zero divisors, so that under our assumptions, RG contains no idempotents.*

Example 2.2.3. *Consider the ring of integers $R = \mathbb{Z}$ and the cyclic group of order five, $G = \langle x \mid x^5 = 1 \rangle$. Define the following elements of RG:*

$$\gamma = 1 - x^2 - x^3, \quad \delta = 1 - x - x^4$$

Exercise 2.2.4. *Verify that $\gamma\delta = 1$. Therefore, the group ring in this example has nontrivial units as well as zero divisors.*

Here is another reason to be interested in whether a group is orderable.

Proposition 2.2.5 ([17])**.** *If G is left-orderable and H is any group, and there is a ring isomorphism $\mathbb{Z}G \cong \mathbb{Z}H$ then there is a group isomorphism $G \cong H$.*

3. Spaces of orderings of a group

We'll consider a natural topology on the set $LO(G)$ of left-orderings of a given group G, and apply it to gain insight into algebraic properties. This concept was introduced in the literature by [26]. We use a slightly different, though equivalent, approach, using the Tychonoff topology on the power set $\mathcal{P}(G)$ of G, a structure familar to point-set topologists.

If $(G, <)$ is a left-ordered group, then the *positive cone*

$$P = P_< = \{g \in G | 1 < g\}$$

is (1) a *semigroup* ($P \cdot P \subset P$) and (2) G is *partitioned* as

$$G = P \sqcup P^{-1} \sqcup \{1\}.$$

Conversely, if a group G has a subset P satisfying (1) and (2), then G can be left-ordered by the rule

$$g < h \Leftrightarrow g^{-1} h \in P$$

Exercise 3.0.1. *Verify that this recipe defines a left order of G. The left-ordering is a* bi-ordering *iff its positive cone is* normal*: for all $g \in G$ we have $g^{-1} P g \subset P$.*

This gives us a one-to-one correspondence between left-orderings and certain subsets of G, that is, elements of the power set $\mathcal{P}(G)$ so we may consider, by abuse of notation, that $LO(G) \subset \mathcal{P}(G)$.

3.1. The Tychonoff topology

We now recall the Tychonoff topology of a cartesian product of (possibly infinitely many) topological spaces. It is the smallest topology such that the projection functions are continuous.

If X is any set, the power set $\mathcal{P}(X)$ can be identified with the set $2^X = \{0,1\}^X$ of all functions $f : X \to \{0,1\}$ via the correspondence of subsets $Y \subset X$ with their characteristic functions $f_Y \in 2^X$ where $f_Y(x) = 1 \iff x \in Y$.

Giving $\{0,1\}$ the discrete topology, $\{0,1\}^X$ is a special case of a product space and can be given the Tychonoff topology. This then defines a topology on $\mathcal{P}(X) \cong \{0,1\}^X$.

Typical open sets in $\mathcal{P}(X)$ are $U_x = \{Y \subset X | x \in Y\}$ and its complement $U_x^c = \{Y \subset X | x \notin Y\}$. Note that these correspond to the sets of functions $\{f \in 2^X \mid f(x) = 1\}$ and $\{f \in 2^X \mid f(x) = 0\}$, respectively.

Finite intersections of such sets form a *basis* for the topology of $\mathcal{P}(X)$; by a theorem of Tychonoff, it is *compact*.

It is also *totally disconnected*: if Y_1 and Y_2 are distinct elements of $\mathcal{P}(X)$, choose an x which is in Y_1 (say) but not in Y_2. Then the sets U_x and U_x^c form a separation of $\mathcal{P}(X)$ with $Y_1 \in U_x$ and $Y_2 \in U_x^c$.

If X is countably infinite, $\mathcal{P}(X)$ is homeomorphic with the Cantor set.

The set $LO(G)$ of left-orderings $<$ of a group G is in this way identified with the set of subsets $P \subset G$ (i. e. elements of $\mathcal{P}(G)$) satisfying

(1) $P \cdot P \subset P$ and

(2) $G = P \sqcup P^{-1} \sqcup \{1\}$

Identifying left-orderings with their positive cones, a basic neighborhood of a left-ordering $<$ of a group G can be defined by considering a *finite* number of inequalities $g_i < h_i$ which hold – the corresponding neighborhood of $<$ is all orderings $\prec$ for which those inequalities remain true: $g_i \prec h_i$.

Proposition 3.1.1. *$LO(G)$ is a closed subset of $\mathcal{P}(G)$.*

Proof: The set of $P \subset G$ which do *not* satisfy (1) is exactly the union over all $g, h \in G$ of the sets $U_g \cap U_h \cap U_{gh}^c$, and is therefore open.

Similarly, one checks that (2) is a closed condition. □

Corollary 3.1.2. *$LO(G)$ is compact and totally disconnected.*

One can also check that the set of all bi-orders of a group G is closed in $LO(G)$, hence also forms a compact, totally disconnected space... Possibly empty!

A basic question regarding a group G is *whether it is left-orderable*, or in other words, is $LO(G)$ *nonempty*? If G is nontrivial, a necessary condition for left-orderability is that G be torsion-free. But that is by no means sufficient; there are many examples of torsion-free groups which are not left-orderable (see Example 4.0.2).

Suppose G is *finitely-generated* with generating set S, and let $B_n(G)$ be the n-ball in the Cayley graph of G with respect to the set S. In other words this is the set of all elements of G which can be written as a product of n or fewer elements of S and their inverses.

Call a subset Q of $B_n(G)$ a *pre-order* if

(1') $(Q \cdot Q) \cap B_n(G) \subset Q$ and

(2') $B_n(G) = Q \sqcup Q^{-1} \sqcup \{1\}$.

If G has a positive cone, then the cone's intersection with $B_n(G)$ is a pre-order for $B_n(G)$, so the following is clear.

Proposition 3.1.3. *If G is left-orderable, then every $B_n(G)$ has a pre-order.*

This is the basis for an algorithm, discussed in [3], to test for left-orderability of a finitely generated group for which the word problem has a solution. Basically it searches expanding balls in the Cayley graph for possible pre-orders, and quits and reports when it finds an n-ball without any pre-order. If the group is left orderable, the algorithm continues forever. Perhaps surprisingly, the converse to Proposition 3.1.3 is also true.

Theorem 3.1.4. *If every $B_n(G), n \geq 1$ has a pre-order, then G is left-orderable.*

Proof: Note that the restriction to $B_n(G)$ of a pre-order for $B_{n+1}(G)$ is a pre-order for $B_n(G)$. Define

$$\mathcal{Q}_n = \{R \subset G | R \cap B_n(G) \text{ is a pre-order for } B_n(G)\}.$$

One checks that $\mathcal{Q}_n$ is a *closed* subset of $\mathcal{P}(G)$ and that $\mathcal{Q}_{n+1} \subset \mathcal{Q}_n$. In a compact space, the *intersection* of a nested sequence of nonempty closed sets is *nonempty*. It is easy to check that a set $P \subset G$ is in every $\mathcal{Q}_n$, if and only if P satisfies (1) and (2). Thus we have

$$LO(G) = \bigcap_{n=1}^{\infty} \mathcal{Q}_n \neq \emptyset.$$

□

The assumption of being finitely-generated is not really essential.

Theorem 3.1.5. *A group is left-orderable if and only each of its finitely-generated subgroups is left-orderable.*

Proof: The forward implication is obvious. For the reverse implication, consider any *finite* subset F of the given group G and let $\langle F \rangle$ denote the subgroup of G generated by F. Define

$$\mathcal{Q}(F) := \{Q \subset G | Q \cap \langle F \rangle \text{ is a positive cone for } \langle F \rangle\}$$

For each finite $F \subset G$, $\mathcal{Q}(F)$ is a *closed nonempty* subset of $\mathcal{P}(G)$.

The family of all $\mathcal{Q}(F)$, for finite $F \subset G$, is a collection of closed sets which has the *finite intersection property*, because

$$\mathcal{Q}(F_1 \cup F_2 \cup \cdots \cup F_n) \subset \mathcal{Q}(F_1) \cap \mathcal{Q}(F_2) \cap \cdots \cap \mathcal{Q}(F_n).$$

By compactness, the entire family must have a nonempty intersection:

$$LO(G) = \bigcap_{F \subset G \text{finite}} \mathcal{Q}(F) \neq \emptyset. \quad \square$$

We note that the same argument may be used to show that a group is bi-orderable if and only if each finitely generated subgroup is bi-orderable.

Corollary 3.1.6. *An abelian group is bi-orderable if and only if it is torsion-free.*

Proof: Bi-orderable groups are torsion-free. To see the other direction, it is enough to observe that a *finitely-generated* torsion-free abelian group is isomorphic with $\mathbb{Z}^n$ for some finite n. □

Theorem 3.1.7. *A group G can be left-ordered if and only if for every finite subset $\{x_1, \ldots, x_n\}$ of $G \setminus \{1\}$, there exist $\epsilon_i = \pm 1$ such that 1 does not belong to the semigroup $S(x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n})$ generated by $x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n}$ in G.*

Proof: One direction is clear, for if $<$ is a left-ordering of G, just choose ϵ_i so that $x_i^{\epsilon_1}$ is greater than the identity. For the converse, we may assume that G is finitely generated, and we need only show that each k-ball $B_k(G)$, with respect to a fixed finite generating set, has a pre-order. Now consider $\{x_1, \ldots, x_n\}$ to be the entire set $B_k(G) \setminus \{1\}$, and choose $\epsilon_i = \pm 1$ such that $1 \notin S(x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n})$. We easily check that the set $\{x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n}\}$ is a pre-order of $B_k(G)$, completing the proof. □

Theorem 3.1.8 (Burns-Hale)**.** *A group G is left-orderable if and only if for every finitely-generated subgroup $H \neq \{1\}$ of G, there exists a left-orderable group L and a nontrivial homomorphism $H \to L$.*

Proof: One direction is obvious. To prove the other direction, assume the subgroup condition. The result will follow if one can show:

Claim: For every finite subset $\{x_1, \ldots, x_n\}$ of $G \setminus \{1\}$, there exist $\epsilon_i = \pm 1$ such that 1 does not belong to $S(x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n})$.

We will establish this claim by induction on n. It is certainly true for $n = 1$, for $S(x_1)$ cannot contain the identity unless x_1 has finite order, which is impossible since the cyclic subgroup $\langle x_1 \rangle$ must map nontrivially to a left-orderable (hence torsion-free) group.

Next assume the claim true for all finite subsets of $G \setminus \{1\}$ having fewer than n elements, and consider $\{x_1, \ldots, x_n\} \subset G \setminus \{1\}$.

By hypothesis, there is a nontrivial homomorphism

$$h : \langle x_1, \ldots, x_n \rangle \to L$$

where $(L, \prec)$ is a left-ordered group. Not all the x_i are in the kernel; we may assume they are numbered so that

$$h(x_i) \begin{cases} \neq 1 \text{ if } i = 1, \ldots, r, \\ = 1 \text{ if } r < i \leq n. \end{cases}$$

Now choose $\epsilon_1, \ldots, \epsilon_r$ so that $1 \prec h(x_i^{\epsilon_i})$ in L for $i = 1, \ldots, r$.

For $i > r$, the induction hypothesis allows us to choose $\epsilon_i = \pm 1$ so that $1 \notin S(x_{r+1}^{\epsilon_{r+1}}, \ldots, x_n^{\epsilon_n})$.

We now check that $1 \notin S(x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n})$ by contradiction. Suppose that 1 *is* a product of some of the $x_i^{\epsilon_i}$. If all the i are greater than r, this is impossible, as $1 \notin S(x_{r+1}^{\epsilon_{r+1}}, \ldots, x_n^{\epsilon_n})$. On the other hand if some i is less than or equal to r, we see that h must send the product to an element strictly greater than the identity in L, again a contradiction. □

Definition 3.1.9. A group is *indicable* if there is a surjection of the group to $\mathbb{Z}$, the infinite cyclic group. A group is *locally indicable* if every nontrivial finitely generated subgroup is indicable.

Corollary 3.1.10. *Locally indicable groups are left-orderable.*

Exercise 3.1.11. *Verify that the properties of being torsion-free, left-orderable or locally indicable are preserved under extensions. That is, if $1 \to K \to G \to H \to 1$ is exact and K and H have the given property, then so does G.*

This is not the case for bi-orderability. The Klein bottle group demonstrates this.

Exercise 3.1.12. *Let $G = \langle x, y | x^{-1}yx = y^{-1}\rangle$ be the fundamental group of the Klein bottle. Let K be the subgroup generated by y.*

Verify that K is normal in G and isomorphic to $\mathbb{Z}$, the group of integers. Moreover $H := G/K$ is also isomorphic to $\mathbb{Z}$.

Therefore we have an exact sequence $1 \to \mathbb{Z} \to G \to \mathbb{Z} \to 1$ and can conclude that G is left-orderable, and in fact locally indicable. Yet it is not bi-orderable, because if it were, the defining relation would imply the contradiction that y is positive if and only if y^{-1} is positive.

4. Knot groups and orderability.

This is the beginning of the second lecture. Recall that we discussed orderability of groups and the closely related concept of local indicability. We have the following implications among these properties:

Theorem 4.0.1. *For a group, the following implications hold: Bi-orderable $\implies$ Locally indicable $\implies$ Left-orderable $\implies$ Torsion-free*

We did not (and won't!) prove the first implication, which is nontrivial and depends on a classical result [13]. Recall that the other implications were proven in the previous lectures.

None of these implications is reversible, as the following examples show. We already observed that the Klein bottle group is locally indicable but not bi-orderable.

An example of a group which is left-orderable but not locally indicable is Example 4.3.1 discussed below. It is the fundamental group of a closed 3-manifold and is finitely generated and left-orderable. On the other hand the manifold has trivial first homology. That implies that the fundamental group is perfect, equal to its commutator subgroup. Any homomorphism from a perfect group to $\mathbb{Z}$ must be trivial.

It remains to give an example of a torsion-free group which is not left-orderable.

Example 4.0.2. *We will consider a crystallographic group G which is torsion-free but not left-orderable. Specifically consider the group G with generators a, b, c acting on $\mathbb{R}^3$ with coordinates (x, y, z) by the rigid motions:*

$$a(x,y,z) = (x+1, 1-y, -z)$$
$$b(x,y,z) = (-x, y+1, 1-z)$$
$$c(x,y,z) = (1-x, -y, z+1)$$

One can easily check the relations $a^2ba^2 = b$, $b^2ab^2 = a$ and $abc = id$. By the last relation we see that one generator may be eliminated. In fact G has the presentation $G = \langle a, b \mid a^2ba^2 = b, b^2ab^2 = a\rangle$.

Exercise 4.0.3. *Check the relations cited above. Argue that the group G is torsion-free.*

Exercise 4.0.4. *Argue that G is not left-orderable as follows. First show that for all choices of $m, n \in \{-1, +1\}$ one has $a^{2m}b^na^{2m} = b^n$ and $b^{2n}a^mb^{2n} = a^m$. Then argue that*

$$\begin{aligned}(a^mb^n)^2(b^na^m)^2 &= a^mb^{-n}b^{2n}a^mb^{2n}a^mb^na^m \\ &= a^mb^{-n}a^{2m}b^na^{2m}a^{-m} \\ &= a^mb^{-n}b^na^{-m} = 1\end{aligned}$$

Conclude that if G were left-orderable, all choices of sign for a and b would lead to a contradiction.

Exercise 4.0.5. *Show that the subgroup $A = \langle a^2, b^2, c^2 \rangle$ is generated by shifts (by even integral amounts) in the directions of the coordinate axes, and so is a free abelian group of rank 3. Moreover A is normal in G and of finite index. Therefore G is virtually bi-orderable, in the sense that a finite index subgroup is bi-orderable.*

4.1. Knot groups and 3-manifold groups

If K is a knot in $\mathbb{S}^3$, its *knot group* is $\pi_1(\mathbb{S}^3 \setminus K)$. Our goal is to show that all knot groups are left-orderable, in fact locally indicable. This will be a special case of a more general result about 3-dimensional manifolds.

We will need a few ideas from 3-manifold theory.

Definition: A 3-manifold is *irreducible* if every tame 2-sphere in the manifold bounds a 3-dimensional ball in the manifold.

A nontrivial fact is that if $\tilde{X} \to X$ is a covering space, with X (and therefore $\tilde{X}$) a 3-manifold, then X is irreducible if and only if $\tilde{X}$ is irreducible [12].

If $X = \mathbb{S}^3 \setminus K$ is a knot complement, then X is irreducible. This is also true if K is a link if (and only if) it is not a split link.

By Alexander duality, we also have that $H_1(\mathbb{S}^3 \setminus K; \mathbb{Z}) \cong \mathbb{Z}$. That is, the first Betti number (the number of copies of $\mathbb{Z}$ appearing in the direct sum decomposition of the first homology group) equals one.

Theorem 4.1.1. *Suppose X is a connected, orientable, irreducible 3-manifold (possibly with boundary). If X has positive first Betti number, then $\pi_1(X)$ is locally indicable, and therefore left-orderable.*

The proof, essentially due to Howie and Short [14], will be given below.

Corollary 4.1.2. *Knot groups are locally indicable.*

Proof: To prove the theorem, consider X as in the hypothesis.

First note that $\pi_1(X)$ is *indicable*, using the (surjective) Hurewicz homomorphism and a further homomorphism to one of the $\mathbb{Z}$ factors of $H_1(X)$.

$$\pi_1(X) \to H_1(X) \to \mathbb{Z}$$

To show $\pi_1(X)$ is *locally* indicable, consider a finitely generated nontrivial subgroup $H < \pi_1(X)$. We need to find a surjection $H \to \mathbb{Z}$.

Case 1: H has *finite index*. This is easy; the Hurewicz map takes H to a finite index subgroup of $H_1(X)$, which therefore maps onto $\mathbb{Z}$.

Case 2: H has *infinite index*. Then there is a covering $p : \tilde{X} \to X$ with $p_* \pi_1(\tilde{X}) = H$. $\tilde{X}$ is noncompact, but its fundamental group is f. g. so, by a theorem of Scott [25], there is a *compact* submanifold $C \subset \tilde{X}$ with inclusion inducing an isomorphism $\pi_1(C) \cong \pi_1(\tilde{X}) \cong H$.

C necessarily has nonempty boundary. If $B \subset \partial C$ is a boundary component which is a sphere, then irreducibility implies that B bounds a 3-ball in $\tilde{X}$. That 3-ball either contains C or its interior is disjoint from C, and the former can't happen because that would imply the inclusion map $\pi_1(C) \to \pi_1(\tilde{X})$ is trivial. Therefore, we can adjoin that 3-ball to C removing B as a boundary component and not changing $\pi_1(C)$. This process allows us to assume that ∂C is nonempty and has infinite homology groups.

Exercise 4.1.3. *Conclude that C also has infinite homology. [Hint: one way to do this is by considering the Euler characteristic of the closed 3-manifold 2C, obtained by glueing two copies of C together along the boundary.]*

Then we have surjections $H \cong \pi_1(C) \to H_1(C) \to \mathbb{Z}$ as required. □

4.2. Fibred knots

It is well known that every (tame) knot in $\mathbb{S}^3$ is the boundary of a compact orientable surface (called a Seifert surface) in $\mathbb{S}^3$.

A knot is said to be *fibred* if there is a fibre bundle map $\mathbb{S}^3 \setminus K \to \mathbb{S}^1$ with fibres being open orientable surfaces each of whose closures has K as boundary.

In other words, the complement of K in $\mathbb{S}^3$ can be filled with a circle's worth of orientable surfaces.

If K is a fibred knot, with complement $X = \mathbb{S}^3 \setminus K$ and with fibre F an open surface, the exact homotopy sequence of a fibration gives the short exact sequence:

$$1 \to \pi_1(F) \to \pi_1(X) \to \pi_1(S^1) \to 1.$$

But $\pi_1(F)$ is a free group and $\pi_1(S^1) \cong \mathbb{Z}$. Both these groups are locally indicable, so we conclude that the knot group $\pi_1(X)$ is locally indicable, and therefore left orderable.

That is, the group of a fibred knot is seen to be locally indicable without the need for the general theorem we have proved, which applies to all knots.

A fibration $X \to S^1$ with fibre F can be considered as the mapping cylinder of a (monodromy) homeomorphism $h : F \to F$:

$$X \cong \frac{F \times [0,1]}{(x,1) \sim (h(x),0)}.$$

For a fibred knot with $X = \mathbb{S}^3 \setminus K$ the Alexander polynomial is just the *characteristic polynomial* of the *homology* monodromy $H_1(F) \to H_1(F)$. Non-fibred knots also have an Alexander polynomial, but it may not be monic, as is the case for fibred knots.

Also, for fibred knots the knot group $\pi_1(X)$ is an HNN extension of the free group $\pi_1(F)$, corresponding to the *homotopy* monodromy

$$h_* : \pi_1(F) \to \pi_1(F),$$

where $\pi_1(F) \cong \langle x_1, \ldots, x_{2g} \rangle$ is a free group. So we have

$$\pi_1(X) \cong \langle x_1, \ldots, x_{2g}, t \mid h_*(x_i) = t x_i t^{-1}, i = 1, \ldots, 2g \rangle.$$

Exercise 4.2.1. *This group is bi-orderable if and only if there is a bi-ordering of* $\pi_1(F)$ *which is preserved by* h_*.

We will sketch the proofs of two theorems regarding bi-ordering fibred knot groups.

Theorem 4.2.2 ([21])**.** *If K is fibred and* $\Delta_K(t)$ *has* all *roots real and positive, then its group* $\pi_1(S^3 \setminus K)$ *is bi-orderable.*

Theorem 4.2.3 ([6])**.** *If K is a nontrivial fibred knot and its group is bi-orderable, then* $\Delta_K(t)$ *has* some *real positive roots.*

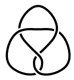
Before proving these theorems, we consider some examples.

Torus knots: curves which can be inscribed on the surface of an unknotted torus in $\mathbb{S}^3$. For relatively prime integers p, q the torus knot $T_{p,q}$ has group

$$\langle a, b | a^p = b^q \rangle.$$

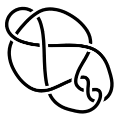
Note that a commutes with b^q but not with b (unless the group is abelian, and the knot unknotted). We've already observed that in a bi-orderable group, if an element commutes with a nonzero power of another element, then the elements must themselves commute. Therefore:

Proposition 4.2.4. *The group of a nontrivial torus knot is not bi-orderable.*

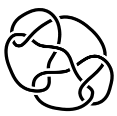
The figure-eight knot 4_1  is fibred and has Alexander polynomial $\Delta_{4_1} = t^2 - 3t + 1$ with roots $\frac{3 \pm \sqrt{5}}{2}$, both real and positive. From Theorem 4.2.2 we conclude

Proposition 4.2.5. *The group of the knot* 4_1 *is bi-orderable.*

More bi-orderable knot groups:

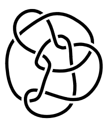


8_{12} $\Delta = 1 - 7t + 13t^2 - 7t^3 + t^4$



10_{137} $\Delta = 1 - 6t + 11t^2 - 6t^3 + t^4$

$11a_5$ $\Delta = 1 - 9t + 30t^2 - 45t^3 + 30t^4 - 9t^5 + t^6$



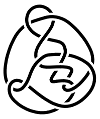
$11n_{142}$ $\Delta = 1 - 8t + 15t^2 - 8t^3 + t^4$

$12a_{0125}$ $\Delta = 1 - 12t + 44t^2 - 67t^3 + 44t^4 - 12t^5 + t^6$

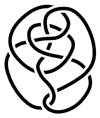
$12a_{0181}$ $\Delta = 1 - 11t + 40t^2 - 61t^3 + 40t^4 - 11t^5 + t^6$

$12a_{1124}$ $\Delta = 1 - 13t + 50t^2 - 77t^3 + 50t^4 - 13t^5 + t^6$

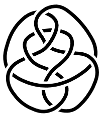
$12n_{0013}$ $\Delta = 1 - 7t + 13t^2 - 7t^3 + t^4$



$12n_{0145}$ $\Delta = 1 - 6t + 11t^2 - 6t^3 + t^4$



$12n_{0462}$ $\Delta = 1 - 6t + 11t^2 - 6t^3 + t^4$



$12n_{0838}$ $\Delta = 1 - 6t + 11t^2 - 6t^3 + t^4$

Recall the Theorem: *fibred and bi-orderable* $\implies$ Δ *has positive roots*.

This can be used for an alternative proof that torus knots $T_{p,q}$, which are fibred, have non-bi-orderable group, because

$$\Delta_{T(p,q)} = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}$$

whose roots are on the unit circle and not real.

There are many other fibred knots which have *non*-biorderable group for similar reasons. Recently, orderability properties have been decided for many non-fibred knots as well; see [5] and [15].

As motivation for the proof of Theorem 4.2.2, consider an upper triangular matrix multiplied by a vector:

$$\begin{pmatrix} \lambda_1 & * & * \\ 0 & \lambda_2 & * \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 x_1 + *x_2 + *x_3 \\ \lambda_2 x_2 + *x_3 \\ \lambda_3 x_3 \end{pmatrix}$$

Now, declaring a vector (in $\mathbb{R}^3$) to be "positive" if its last nonzero entry is greater than zero, we see that, if also the eigenvectors λ_i are positive, then multiplication by such a matrix preserves that positive cone of $\mathbb{R}^3$, considered as an additive group. This generalizes to the following:

Proposition 4.2.6. *If all the eigenvalues of a linear transformation* $L : \mathbb{R}^n \to \mathbb{R}^n$ *are real and positive, then there is a bi-ordering of* $\mathbb{R}^n$ *which is preserved by L.*

So our problem reduces to showing:

Proposition 4.2.7. *Let F be a finitely generated free group and* $h : F \to F$ *an automorphism. If all the eigenvalues of* $h_* : H_1(F;\mathbb{Q}) \to H_1(F;\mathbb{Q})$ *are real and positive, then there is a bi-ordering of F preserved by h.*

Proof: One way to order a free group F is to use the lower central series $\gamma_0(F) \supset \gamma_1(F) \supset \cdots$ defined by

$$\gamma_0(F) = F, \quad \gamma_{i+1}(F) = [F, \gamma_i(F)].$$

These are all normal subgroups and for free groups have the properties that $\cap_{i=0}^{\infty}\gamma_i(F) = \{1\}$ and all the quotients $\gamma_i(F)/\gamma_{i+1}(F)$ are finitely generated free abelian groups. One can choose an arbitrary ordering of each quotient $\gamma_i(F)/\gamma_{i+1}(F)$ and declare $1 \neq x \in F$ to be positive if its class in $\gamma_i(F)/\gamma_{i+1}(F)$ is positive in the chosen ordering, where i is the greatest subscript such that $x \in F_i$. It's quite routine to verify that this defines a bi-ordering of F.

If $h : F \to F$ is an automorphism it preserves the lower central series and therefore induces maps of the lower central quotients:

$$h_i : \gamma_i(F)/\gamma_{i+1}(F) \to \gamma_i(F)/\gamma_{i+1}(F).$$

With this notation, h_0 is just the abelianization h_{ab}. In a sense, all the h_i are determined by h_0. Specifically, there is an embedding of $\gamma_i(F)/\gamma_{i+1}(F)$ in the tensor power $F_{ab}^{\otimes i}$, and the map h_i is just the restriction of $h_{ab}^{\otimes i}$.

The assumption that all eigenvalues of h_{ab} are real and positive implies that the same is true of all its tensor powers. This allows us to find bi-orderings of the free abelian groups $\gamma_i(F)/\gamma_{i+1}(F)$ which are invariant under the h_i according to Proposition 4.2.6. Using these to bi-order F, we get invariance under h, which proves the proposition and the theorem. □

We now turn to the proof of Theorem 4.2.3 that If K is fibred and its group is bi-orderable, then $\Delta_K(t)$ has *some* real positive roots. That result follows from this more general algebraic result:

Theorem 4.2.8. *Suppose G is a nontrivial finitely generated bi-orderable group and that $\phi : G \to G$ preserves a bi-ordering of G. Then the induced map*

$$\phi_* : H_1(G;\mathbb{Q}) \to H_1(G;\mathbb{Q})$$

has a positive eigenvalue.

Proof: The key idea is to consider a linear automorphism $L : \mathbb{Q}^n \to \mathbb{Q}^n$ which preserves an ordering. Regarding $\mathbb{Q}^n$ as a subset of $\mathbb{R}^n$, there is a hyperplane $H \subset \mathbb{R}^n$ defined by

$H = \{x \in \mathbb{R}^n |$ *every nbhd. of x contains positive and negative points*$\}$

One easily checks that H is a linear subspace, that H separates $\mathbb{R}^n$ (and hence has codimension 1) and is invariant under L.

Consider the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$, and let D denote the closed hemisphere of $\mathbb{S}^{n-1}$ which lies on the “positive" side of H. There is a mapping $D \to D$ defined by

$$x \to \frac{L(x)}{|L(x)|}.$$

Since D is an $(n-1)$-ball, this map has a *fixed point* (Brouwer). This fixed point corresponds to an eigenvector of L, which has a positive real eigenvalue. □

4.3. Surgery

Now let’s consider some applications to *surgery* on a knot K in $\mathbb{S}^3$. In our context, surgery means that one removes a tubular neighborhood of K and attaches a solid torus $\mathbb{S}^1 \times D^2$ so that the meridian $\{*\} \times \mathbb{S}^1$ is attached to a specified “framing” curve on the boundary of the neighborhood.

By theorem of Wallace and Lickorish, every compact, orientable 3-manifold (without boundary) can be constructed by surgery on some disjoint union of knots (i. e. a link) in $\mathbb{S}^3$.

Consider surgery on the trefoil knot as depicted in Figure 4.3

With +1 framing, as pictured, one gets the Poincaré homology sphere, as constructed by Max Dehn [7]. This is a 3-manifold with the same homology as $\mathbb{S}^3$, with fundamental group

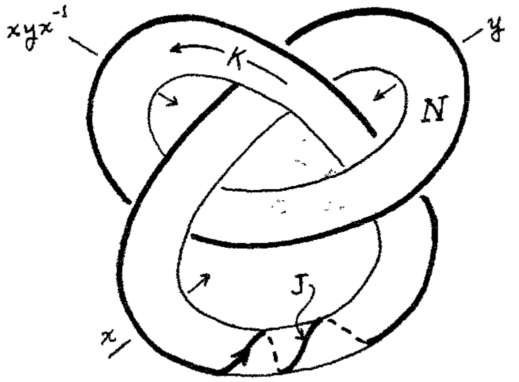


Figure 4.1: Surgery on the trefoil with coefficient +1

$$\langle a, b | (ab)^2 = a^3 = b^5 \rangle$$

This is a finite group, of order 120, so its group is certainly *not* left-orderable.

For the next example, we'll need to consider $\widetilde{SL_2}(\mathbb{R})$, which is one of the eight Thurston 3-manifold geometries, as we've already discussed.

Example 4.3.1. *If we do surgery on the trefoil using -1 framing, the resulting 3-manifold M, again a homology sphere, has fundamental group*

$$\langle a, b | (ab)^2 = a^3 = b^7 \rangle.$$

G. Bergman [2] *observed that this group maps injectively to* $\widetilde{SL_2}(\mathbb{R})$*, which is a left-orderable group. Thus* $\pi_1(M)$ is *left-orderable (even though its first Betti number is zero).*

It is not bi-orderable or even locally indicable, because it is finitely-generated and perfect (that is, abelianizes to the trivial group).

In [6] we show the following two theorems concerning surgery and orderability.

Theorem 4.3.2. *Suppose K is a fibred knot in* S^3 *and nontrivial surgery on K produces a 3-manifold M whose fundamental group is* bi-orderable*. Then the surgery must be longitudinal (that is, 0-framed) and* $\Delta_K(t)$ *must have a positive real root. Moreover, M fibres over* S^1*.*

In [20] an *L-space* is defined to be a closed 3-manifold M such that $H_1(M;\mathbb{Q}) = 0$ and its Heegaard-Floer homology $\widehat{HF}(M)$ is a free abelian group of rank equal to $|H_1(M;\mathbb{Z})|$. Lens spaces, and more generally 3-manifolds with finite fundamental group are examples of L-spaces. But there are also many L-spaces whose fundamental group is infinite.

Theorem 4.3.3. *Suppose* $K \subset S^3$ *is a knot whose group is bi-orderable. Then one cannot obtain an L-space by surgery on K.*

Proof sketch: Suppose surgery on K results in an L-space. By Yi Ni [19], K must be fibred. Moreover, Ozsváth and Szabó show that the Alexander polynomial of K must have a special form. Then one argues that a polynomial of this form has no positive real roots, so the knot group cannot be bi-ordered (see [6] for details). □

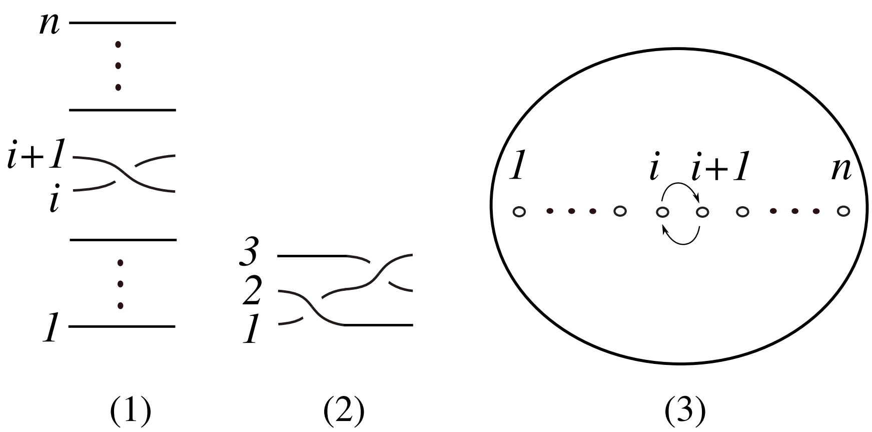


Figure 5.1: (1) pictures the braid $\sigma_i \in B_n$. (2) is the 3-braid $\sigma_1\sigma_2^{-1}$ (3) shows the action of σ_i on the mapping class group of the n-punctured disk.

5. Braids, $Aut(F_n)$ and hyperbolic 3-manifolds

This is the beginning of the third lecture, and represents joint work with Eiko Kin, Osaka University. Much of this will appear soon in [16]. The central theme of today's talk is the interplay between braids and bi-orderings of free groups. We'll see that this also has connections with orderability of certain link groups, and application to understanding minimal volume cusped hyperbolic 3-manifolds.

5.1. Braids and automorphisms

You are all familiar with the braid groups B_n, so I'll just review a few things to make my conventions clear. We think of braids as strings or equivalently certain homeomorphisms of a punctured disk, in both cases up to a natural equivalence. Figure 2 illustrates the standard Artin generators $\sigma_1, \ldots, \sigma_{n-1}$ of B_n.

B_n acts on the fundamental group of the punctured disk, which is the free group F_n. The Artin action of σ_i on $F_n \cong \langle x_1, \ldots, x_n \rangle$ is

$$x_i \to x_i x_{i+1} x_i^{-1} \quad x_{i+1} \to x_i \quad x_j \to x_j,\ j \neq i, i+1$$

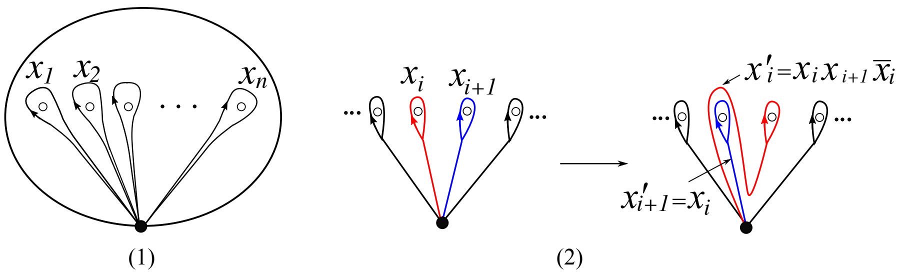


Figure 5.2: Mapping class of the disk corresponding to σ_i.

The Artin representation is an *injective* homomorphism

$$B_n \to Aut(F_n).$$

Because we customarily read braid words from left to right, we'll consider braids to act on the right and use the notation

$$x \to x^\beta$$

to denote the action of the braid β upon the group element $x \in F_n$ under this representation. Note that $x^{\beta\gamma} = (x^\beta)^\gamma$.

We say the braid $\beta \in B_n$ is *order preserving* if and only if there *exists* a bi-ordering $<$ of F_n such that

$$x < y \iff x^\beta < y^\beta$$

We begin with some easy observations regarding order-preserving braids.

Proposition 5.1.1. *The braid σ_i is not order-preserving.*

To see this, recall that σ_i acts by $x_{i+1} \to x_i \to x_i x_{i+1} x_i^{-1}$.

If $<$ is a supposed invariant bi-ordering of F_n, we may assume w.l.o.g. that $x_i < x_{i+1}$. Then, by invariance, $x_i x_{i+1} x_i^{-1} < x_i$. Since bi-orderings are invariant under conjugation we conclude that $x_{i+1} < x_i$, a contradiction. □

Proposition 5.1.2. *The full-twist n-braid $\Delta^2 = (\sigma_1\sigma_2 \cdots \sigma_{n-1})^n$ is order-preserving. In fact its action preserves every bi-ordering of F_n.*

Proof: That's because Δ^2 acts on F_n by conjugation, by $x_1 x_2 \ldots x_n$. And every bi-ordering is invariant under conjugation. □

Recall in the proof of Proposition 4.2.7 we ordered free groups by considering their lower central quotients. Orderings constructed as we've described in the proof of Proposition 4.2.7 will be called *LCS-type* orderings.

Note that any automorphism $\phi : F_n \to F_n$ takes each lower central subgroup into itself, so ϕ induces homomorphisms

$$\phi_k : \gamma_k(F_n)/\gamma_{k+1}(F_n) \to \gamma_k(F_n)/\gamma_{k+1}(F_n).$$

The homomorphism ϕ_0 is just the abelianization $\phi_{ab} : \mathbb{Z}^n \to \mathbb{Z}^n$.

The key to the following theorem is an algebraic lemma.

Lemma 5.1.3. *Suppose G is a group, $\phi : G \to G$ is an automorphism, and $\phi_i : \gamma_i(G)/\gamma_{i+1}(G) \to \gamma_i(G)/\gamma_{i+1}(G)$ are the induced mappings. If ϕ_0 is the identity mapping, then so is every ϕ_i.*

Exercise 5.1.4. *Prove this lemma using induction. Show that it implies the following theorem.*

Theorem 5.1.5. *Suppose that $\phi : F_n \to F_n$ is an automorphism and that $\phi_{ab} : \mathbb{Z}^n \to \mathbb{Z}^n$ is the* identity *mapping. Then $\phi : F_n \to F_n$ preserves* every *ordering of LCS-type.*

Recall that a *pure* braid is one whose underlying permutation is the identity. The pure braids form a normal subgroup of B_n of index $n!$.

Under the Artin representation, a pure braid sends each generator to some conjugate of itself. Such an automorphism abelianizes to the identity.

Corollary 5.1.6. *If $\beta \in B_n$ is a* pure *braid, then β is order-preserving. In fact, β preserves* every *ordering of F_n of LCS-type.*

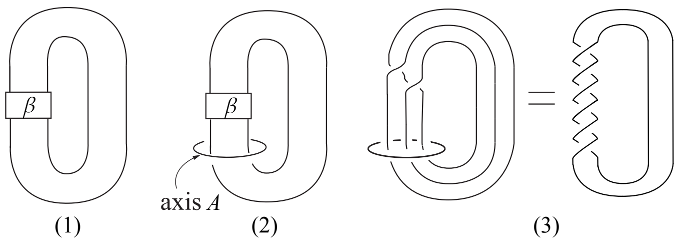


Figure 5.3: (1) Closure $\widehat{\beta}$. (2) br(β) = $\widehat{\beta} \cup A$. (3) br($\sigma_1\sigma_2$) is equivalent to the $(6, 2)$-torus link.

We recall the HNN extension of a group K associated with an automorphism $\phi : K \to K$. If $k_1, \ldots, k_n$ are generators, we introduce a new symbol t and impose the relations

$$t^{-1}k_i t = k_i^{\phi}$$

Denote the resulting group $G = K \rtimes_\phi \mathbb{Z}$.

The following was Exercise 4.2.1.

Proposition 5.1.7. *Suppose K is bi-orderable and $\phi : K \to K$ is an automorphism. Then $G = K \rtimes_\phi \mathbb{Z}$ is bi-orderable if and only if there exists a bi-ordering of K which is preserved by ϕ.*

An example of this is the fundamental group of a fibre bundle over the circle. If $h : X \to X$ is a homeomorphism of the space X, then the *mapping torus* is the space

$$\mathbb{T}_h := X \times [0, 1]/(x, 1) \sim (h(x), 0).$$

There is a natural fibration $\mathbb{T}_h \to S^1$, with fibre X.

The fundamental group of the mapping torus is the extension

$$\pi_1(\mathbb{T}_h) \cong \pi_1(X) \rtimes_{h_*} \mathbb{Z}$$

where $h_* : \pi_1(X) \to \pi_1(X)$ is the ‘homotopy monodromy.’

Back to the situation of braids, any braid $\beta \in B_n$ gives rise to a knot or link $\hat{\beta}$ in the 3-sphere S^3, as in Figure 5.3 (1). We also consider the “braided link” $br(\beta) = \hat{\beta} \cup A$ consisting of $\hat{\beta}$ together with the braid axis, as depicted in Figure 5.3 (2).

Recall that a braid $\beta \in B_n$ acts on the punctured disk D_n. The mapping torus of this action is homeomorphic with the complement of the braided link $br(\beta) = \hat{\beta} \cup A$:

$$\mathbb{T}_\beta \cong S^3 \setminus br(\beta).$$

Proposition 5.1.7 implies the following.

Proposition 5.1.8. *For braid $\beta \in B_n$ the following are equivalent:*

- *β is order preserving,*
- *the fundamental group of $\mathbb{T}_\beta$ is bi-orderable*
- *the link group $\pi_1(S^3 \setminus br(\beta))$ is bi-orderable.*

Proposition 5.1.9. *If a braid $\beta \in B_n$ is order-preserving, then so are all its powers and all its conjugates.*

The proof of this is left as an easy exercise. Also, since every braid has some power which is a pure braid, Corollary 5.1.6 implies the following.

Proposition 5.1.10. *For any braid β some power β^k is order-preserving.*

We call a braid $\beta \in B_n$ *periodic* if some power β^k lies in the centre of B_n. Recall that, for $n \geq 3$ the centre of B_n is infinite cyclic, generated by the full twist Δ_n^2.

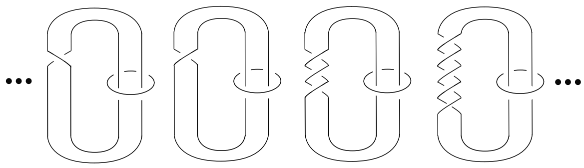


Figure 5.4: Links whose groups are not bi-orderable.

Notice that the links in Figure 5.4 all have homeomorphic complements, by applying disk twists, as described below. Similarly for the next figure.

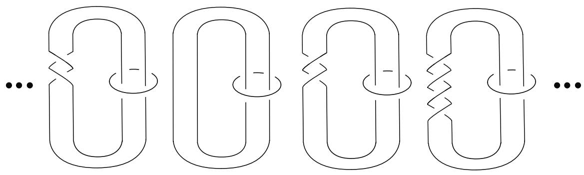


Figure 5.5: Links whose groups are bi-orderable.

Define $\delta_n = \sigma_1\sigma_2 \cdots \sigma_{n-1}$. Noting that $\delta_n^n = \Delta_n^2$ we see that δ_n is periodic, being an n^{th} root of a full twist. There is also an $n-1$ root of a full twist, namely $\delta_n\sigma_1$.

The following result was proved in [11].

Proposition 5.1.11. *Every periodic braid is conjugate to a power of δ_n or $\delta_n\sigma_1$.*

Theorem 5.1.12. *Let $\beta \in B_n$ be a periodic braid. If β is conjugate to $(\delta_n\sigma_1)^k$ then β is order-preserving. If β is conjugate to δ_n^k then β is NOT order-preserving, unless $k \equiv 0 \pmod{n}$.*

Proof: Part of this theorem can be seen using a trick: a *disk twist*, which is a self-homeomorphism of the complement of an unknotted component of a link. Since the complement of the unknot is an open solid torus it has a self-homeomorphism which takes a meridian of the unknot to meridian plus longitude. One can think of this as taking a thickening $D^2 \times [0,1]$ of a disk bounded by the unknotted component, and applying the homeomorphism

$$(z,t) \to (ze^{2\pi t i}, t), \quad z \in D^2 \subset \mathbb{C}, \quad 0 \leq t \leq 1.$$

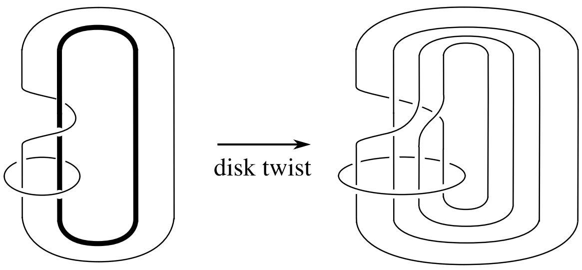


Figure 5.6: nth power of the disk twist converts the braided link of σ_1^2 to that of $\sigma_1\sigma_2\cdots\sigma_{n+1}\sigma_1$. ($n = 2$ in this case.)

Note that in Figure 5.6 the link on the right and the link on the left have *homeomorphic complements*. The braid on the right is pure, therefore order-preserving, and so the complement of its braided link has bi-orderable group (in fact isomorphic to a direct product $F_2 \times \mathbb{Z}$. We conclude that (in this picture) $\delta_4\sigma_1$ must be order-preserving. We see, in fact, that the link group of $br(\delta_n\sigma_1)$ is $F_2 \times \mathbb{Z}$, We can also observe that for $k > 1$ there is a k-fold covering space $br((\delta_n\sigma_1)^k) \to br(\delta_n\sigma_1)$, and the group $br((\delta_n\sigma_1)^k)$, being a subgroup, is therefore also bi-orderable. □

5.2. Tensor product of braids.

Given braids $\alpha \in B_m$ and $\beta \in B_n$ we can form the tensor product $\alpha \otimes \beta \in B_{m+n}$ as shown in Figure 5.7 .

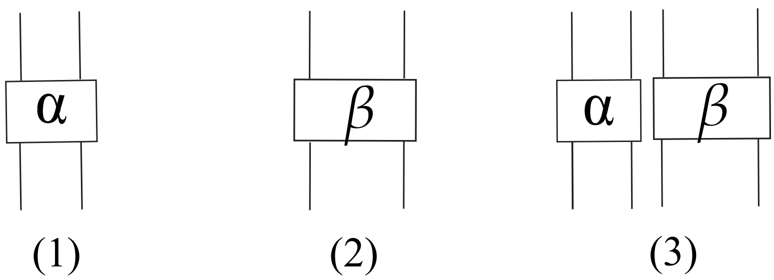


Figure 5.7: (1) $\alpha \in B_m$. (2) $\beta \in B_n$. (3) $\alpha \otimes \beta \in B_{m+n}$.

Proposition 5.2.1. *The braid $\alpha \otimes \beta$ is order-preserving if and only if both α and β are order-preserving.*

This is a consequence of the following recent result of mine [22].

Theorem 5.2.2. *Suppose $(G, <_G)$ and $(H, <_H)$ are bi-ordered groups. Then there is a bi-ordering of $G * H$ which extends the orderings of the factors and such that whenever $\phi : G \to G$ and $\psi : H \to H$ are order-preserving automorphisms, the ordering of $G * H$ is preserved by the automorphism $\phi * \psi : G * H \to G * H$.*

Corollary 5.2.3. *A braid $\beta \in B_m$ is order-preserving if and only if $\beta \otimes 1_n \in B_{m+n}$ is order-preserving.*

Note that the order-preserving braids in B_2 are exactly the powers σ_1^k with k even. In other words, it is the subgroup of pure 2-braids.

For $n > 2$ the situation is different.

Proposition 5.2.4. *For $n > 2$, the set of order-preserving braids is NOT a subgroup of B_n.*

Proof: Consider $\alpha = \sigma_1\sigma_2\sigma_1$, which is (an extension of) the periodic braid $\delta_2\sigma_1 \in B_3$, hence order-preserving. Let $\beta = \sigma_1^{-2}$, a pure braid, so also order-preserving. But the product $\alpha\beta = \sigma_1\sigma_2\sigma_1^{-1}$ is not order-preserving, as it is conjugate to σ_2 which is not order-preserving. □

We saw that the set of order-preserving braids in B_n contains the index $n!$ subgroup of pure braids. But it contains more:

Proposition 5.2.5. *For $n > 2$, the set of order-preserving braids in B_n generates B_n.*

Proof: To see this, note that the above argument shows how to express σ_2 as a product of order-preserving braids. Since all the standard braid generators are conjugate, this tells us how to express any σ_i as a product of order-preserving braids. □

6. Small hyperbolic 3-manifolds

We now turn attention to applications to understanding minimal volume hyperbolic manifolds, possibly with cusps. Recalling Proposition 4.2.7 and Theorem 4.2.8, which may be combined as follows.

Theorem 6.0.1. *Let $\phi : F_n \to F_n$ be an automorphism of the finitely generated free group F_n. If every eigenvalue of $\phi_{ab} : \mathbb{Z}^n \to \mathbb{Z}^n$ is real and positive, then there is a bi-ordering of F_n which is ϕ-invariant. If there exists a bi-ordering of F_n which is ϕ-invariant, then ϕ_{ab} has at least one real and positive eigenvalue.*

Theorem 6.0.2 ([10])**.** *The (unique) minimal volume closed hyperbolic 3-manifold is the Fomenko-Matveev-Weeks manifold, which can be obtained from the Whitehead link by* $[5/2, 5/1]$ *surgery.*

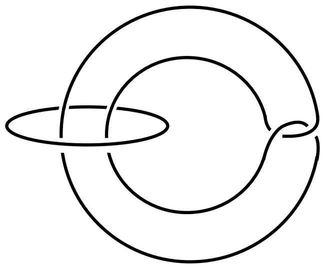


Figure 6.1: The Whitehead link, which produces the Fomenko-Matveev-Weeks manifold

Theorem 6.0.3 ([3])**.** *The fundamental group of the Weeks manifold is NOT left-orderable.*

In the case of *one cusp*, there are two distinct examples.

Theorem 6.0.4 ([4])**.** *A minimal volume one-cusped orientable hyperbolic 3-manifold is homeomorphic to either the complement of the figure-eight knot 4_1, or its sibling, which can be described as* 5/1 *surgery on one component of the Whitehead link.*

The following shows they can be distinguished by orderability properties of their fundamental groups.

Theorem 6.0.5. *The figure-eight complement has bi-orderable fundamental group. The group of its sibling is NOT bi-orderable.*

Proof: To see this, we note that both these manifolds can be realized as punctured torus bundles over S^1. In the case of the figure-eight complement, the (homology) monodromy $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ has two positive eigenvalues $(3 \pm \sqrt{5})/2$. Thus the homotopy monodromy preserves an ordering of F_2, the fundamental group of the fibre, and therefore the mapping torus $S^3 \setminus 4_1$ has bi-orderable group.

The sibling has the monodromy $\begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}$. This has the two negative eigenvalues $(-3 \pm \sqrt{5})/2$. Therefore the homotopy monodromy cannot preserve a bi-order and so its mapping torus (the sibling) has NON-bi-orderable group. □

Theorem 6.0.6 ([1])**.** *A minimal volume orientable hyperbolic* 3*-manifold with* 2 *cusps is homeomorphic to either the Whitehead link complement or the* $(-2, 3, 8)$*-pretzel link complement.*

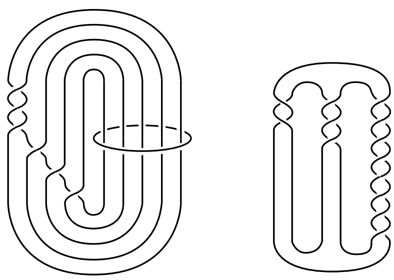


Figure 6.2: Two pictures of the $(-2, 3, 8)$-pretzel link. On the left, we may consider it the braided link $br(\sigma_1^2 \delta_5)$

.

Theorem 6.0.7. *The fundamental group of the Whitehead link complement is bi-orderable. The group of the* $(-2, 3, 8)$*-pretzel link is NOT bi-orderable.*

For the Whitehead link, whose complement fibres over S^1 with fibre a twice-punctured torus, one computes the homology monodromy $\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, which has 1 as a triple eigenvalue. Therefore the homotopy monodromy preserves a bi-order of F_3, and so the group of the Whitehead link is bi-orderable.

For the $(-2, 3, 8)$-pretzel, which is also $br(\delta_5 \sigma_1^2)$, we conclude that its group cannot be bi-ordered by the observation:

Proposition 6.0.8. *For any* $n \geq 3$ *and positive integer* k*, the braid* $\delta_n \sigma_1^{2k}$ *is not order-preserving.*

This can be proved by calculating the action of $\delta_n \sigma_1^{2k}$ on F_n, assuming it is order-preserving, and arriving at a contradiction.

It is conjectured [1] that for 3, 4, 5, 6 cusps, and perhaps up to ten, a minimal volume orientable hyperbolic manifold is homeomorphic with the complement of a “minimally twisted" chain link as pictured in Figure 6.3. The case of 4 cusps was recently settled.

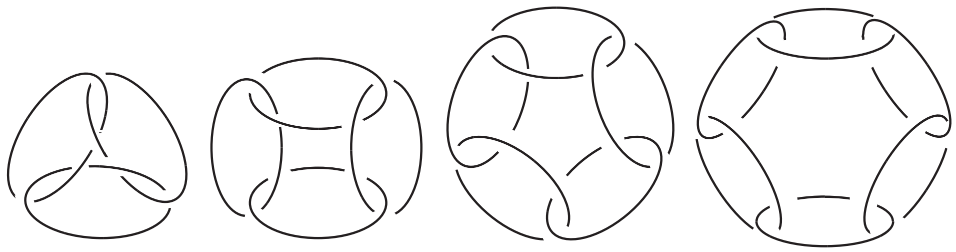


Figure 6.3: (1) C_3. (2) C_4. (3) C_5. (4) C_6.

Theorem 6.0.9 ([28]). *A minimal volume orientable hyperbolic* 3*-manifold with* 4 *cusps is homeomorphic to* $S^3 \setminus C_4$*.*

Theorem 6.0.10. $\pi_1(S^3 \setminus C_4)$ *is bi-orderable.*

Proof: To prove this, we note that $S^3 \setminus C_4$ is homeomorphic (via a disk twist) to $br(\sigma_1^{-2}\sigma_2^2)$. The braid $\sigma_1^{-2}\sigma_2^2$ is order-preserving, as it is a pure braid. □

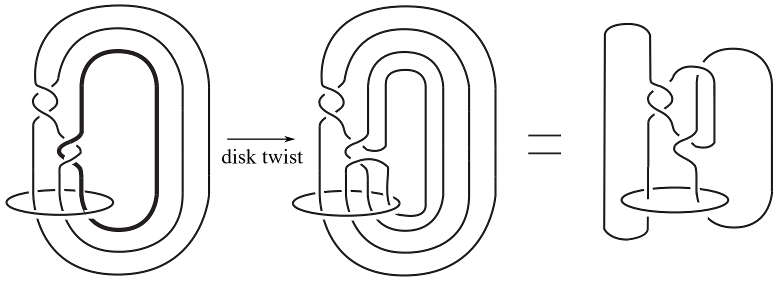


Figure 6.4: $S^3 \setminus \mathrm{br}(\sigma_1^{-2}\sigma_2^2)$ and $S^3 \setminus C_4$ are homeomorphic. (1) $\mathrm{br}(\sigma_1^{-2}\sigma_2^2)$. (2)(3) Links which are equivalent to C_4.

For five cusps, the complement of the minimally twisted 5-chain is conjectured to be minimal among 5-cusped orientable hyperbolic manifolds.

Theorem 6.0.11. $\pi_1(S^3 \setminus C_5)$ *is bi-orderable.*

Proof: This follows from the observation in Figure 6.5 that the complement of C_5 and the complement of the braided link $\mathrm{br}(\sigma_1^{-2}\sigma_2^{-2}\sigma_3^{-2})$ are homeomorphic. This has bi-orderable group as the braid is a pure braid. □

A similar construction can be used to establish the following.

Theorem 6.0.12. $\pi_1(S^3 \setminus C_6)$ *is bi-orderable.*

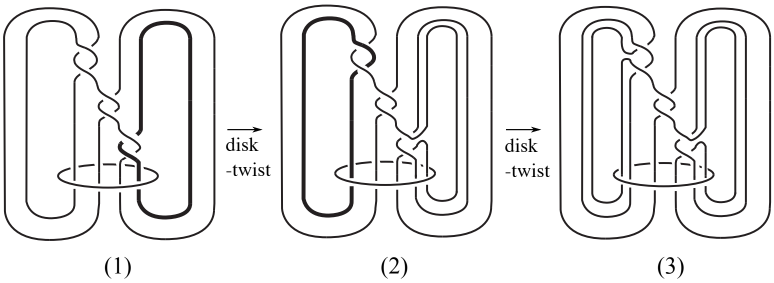


Figure 6.5: $S^3 \setminus \mathrm{br}(\sigma_1^{-2}\sigma_2^{-2}\sigma_3^{-2})$ and $S^3 \setminus C_5$ are homeomorphic. (1) $\mathrm{br}(\sigma_1^{-2}\sigma_2^{-2}\sigma_3^{-2})$. (3) Link which is equivalent to C_5.

As already mentioned, $S^3 \setminus C_6$ is conjectured to be minimal among 6-cusped examples. Similarly, $S^3 \setminus C_3$, a.k.a. the “magic manifold,” is conjectured to be minimal among 3-cusped orientable hyperbolic 3-manifolds. We do not know if its fundamental group is bi-orderable. One can realize $S^3 \setminus C_3$ as the complement of $br(\sigma_1^2\sigma_2^{-1})$. So we'll conclude with an open question.

Question: Is $\sigma_1^2\sigma_2^{-1} \in B_3$ order-preserving?

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